BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

FELIX BROWDER E. E. MOISE

WALTER RUDIN

VOLUME 67, NUMBER 3

MAY, 1961

(WHOLE NO. 672)

CONTENTS

PUBLISHED BY THE SOCIETY

MENASHA, WISCONSIN, AND PROVIDENCE, RHODE ISLAND

The annual dues of members of the Society (see this Bulletin, November, 1958, pp. 405–406) are payable to the Treasurer of the Society, A. E. Meder, Jr., 190 Hope Street, Providence 6, R. I.

The Bulletin of the American Mathematical Society is published bimonthly, in January, March, May, July, September, and November. Price per annual volume is $7.00. Subscriptions, orders for back numbers, and inquiries in regard to nondelivery of current numbers should be addressed to the American Mathematical Society, Curtis Reed Plaza, Menasha, Wis., or 190 Hope Street, Providence 6, R. I.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at the special rate of postage provided for in the act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R., authorized May 9, 1935.

BOOK REVIEWS

Éléments de géométrie algébrique. Par A. Grothendieck, rédigés avec la collaboration de J. Dieudonné. Publications de l'Institut des Hautes Études Scientifiques No. 4, Paris, 1960. 228 pp. 27 NF.

The present work, of which Chapters 0 and I are now appearing together, is one of the major landmarks in the development of algebraic geometry. It plans to cover eventually everything that is known in algebraic geometry over arbitrary ground rings, and of course a lot more besides. A tentative list of its chapters is as follows:

Chapter I. Le langage des schémas.
II. Étude globale élémentaire de quelques classes de morphismes.
III. Cohomologie des faisceaux algébriques cohérents. Applications.
IV. Étude locale des morphismes.
V. Procédés élémentaires de construction de schémas.
VI. Technique de descente. Méthode générale de construction de schémas.
VII. Schémas de groupes, espaces fibrés principaux.
VIII. Étude differentielle des espaces fibrés.
IX. Le groupe fondamental.
X. Résidus et dualité.
XI. Théories d'intersection, classes de Chern, théorème de Riemann-Roch.
XII. Schémas abeliens et schémas de Picard.
XIII. Cohomologie de Weil.

The list is subject to modifications, especially in so far as later chapters are concerned, partly because much of the research needed to complete these chapters remains to be done.

To give the prospective reader some idea of the size of the work, suffice it to say that Chapter I is 134 pages long, that subsequent chapters are expected to be at least as long (probably around 150 pages each), that all chapters are regarded as being open (i.e., subject to additions such as are deemed necessary in the course of the writing), and that Chapters 0 and I together weigh 1 and 3/4 pounds in their present form.

In order to get a more specific idea of what is to come, one should consult first Grothendieck's address to the International Congress at

Edinburgh, 1958, and also the whole series of talks at Bourbaki seminars given in the past two years (available at the Institut Henri Poincaré, 11 Rue Pierre Curie, Paris) in which he has given a sketch of the proofs of important results to appear in later chapters. These talks will provide the necessary motivation to the whole work. They are written concisely, directly, and excitingly. Such motivation could not be given in the actual text, which is written very lucidly, is perfectly organized, and very precise. Thanks are due here to Dieudonné, without whose collaboration the labor involved in writing and publishing the work would have been insurmountable.

Before we go into a closer description of the contents of Chapters 0 and I, it is necessary to say a few words explaining why the present treatise differs radically in its point of view from previous ones.

1. Most of algebraic geometry up to now has been concerned with varieties, say over arbitrary fields. It includes some results on algebraic families of varieties, but such results are few in number, and it has become increasingly clear in recent years that one was facing serious difficulties in dealing with such algebraic systems. For example, the geometer is able to attach to a fixed variety other geometric objects, say a Picard variety. It is then a problem to show that if one has an algebraic system of varieties, the Picard varieties can be associated in such a way that they move along with the varieties, following the same parameter variety, even when special members of the family are degenerate. The tools available at present to deal with such a problem are recognized to be deficient (although of course in special cases, interesting results have been obtained, especially for non-degenerate fibers).

2. In applications to number theory, it has been realized for some time that the reduction mod $\mathfrak{p}$ of a variety defined over a number field was completely analogous to the situation of algebraic systems, a fiber being such a reduction. Although it was possible here again to give an ad hoc definition and results having useful applications to interesting special problems, the theory was technically disagreeable to apply, to say the least.

In order to deal efficiently with the above two points, it was necessary to incorporate from the start into the foundations the notion of a variety defined over a ring, not necessarily Noetherian, and having nilpotent elements (say to reduce mod $\mathfrak{p}^n$, or to describe degenerate fibers in a system). This meant that a variety could not be regarded any more as a model of a "function field," and thus that it should be defined starting with a local description supplemented by a method for gluing local pieces together (sheaves being the natural tool here).

3. The classical tools available were impotent to deal with the problem of defining the homology and homotopy functors to which one is accustomed in topology, and having similar properties. The necessity of having the homology functor, say, was made clear by Weil, who pointed out that if one has it, then the structure of the zeta function for non-singular projective varieties defined over finite fields follows immediately from the Lefschetz fixed point formula. In order to have this, a minimum requirement is that the homology groups H_n associated with a variety V be modules, or vector spaces having characteristic 0 (no matter what the characteristic of the field of definition of V is!).

4. The study and classification of non-abelian coverings of varieties, and in particular the determination of the fundamental group, was completely outside the range of available methods, except for varieties defined over the complex numbers where one could use transcendental methods.

The above list could be expanded, but it gives a good idea why a new approach to algebraic geometry was needed.

Let us now give a closer look at the contents of Chapters 0 and I.

Chapter 0 is intended to include results of commutative algebra needed for the geometric applications. They are more or less well known, but it is difficult to give references for them. The reader should skip this chapter until he meets a place where he needs it. He should start reading Chapter I immediately. For this, he needs to know only what a ring is (commutativity and unit element are always assumed), and the definition of a ring of fractions, which runs as follows. Let A be a ring, S a subset of A closed under multiplication and containing 1. One considers equivalence classes of pairs (a, s) with $a \in A$ and $s \in S$ such that $(a, s) \sim (a', s')$ if there exists $s_1 \in S$ such that $s_1(s'a - sa') = 0$. The equivalence class of (a, s) is denoted by a/s, and these form a ring in the obvious way. This ring is denoted by $S^{-1}A$, and is 0 if S contains nilpotent elements. The most important case is that where S is the complement of a prime ideal $\mathfrak{p}$, so that $S^{-1}A = A_{\mathfrak{p}}$ is the local ring at $\mathfrak{p}$.

We recall that a ringed space is a pair (X, O_X) consisting of a topological space X and a sheaf of rings O_X. Ringed spaces form a category: A morphism $(X, O_X) \to (Y, O_Y)$ is a pair consisting of a continuous map $\phi: X \to Y$ and a contravariant map $\psi: O_Y \to O_X$ compatible with f. If we denote by O_x the fiber of O_X above a point $x \in X$, then ψ induces a homomorphism $\psi_x: O_{\phi(x)} \to O_x$. The ringed space (X, O_X) is called a local ringed space if all the rings O_x are local rings. If (X, O_X) and (Y, O_Y) are local ringed spaces, a morphism (ϕ, ψ) above is called local if the inverse image of the maximal ideal of O_x by ψ_x is the maximal ideal of $O_{\phi(x)}$. The local ringed spaces and the local morphisms then form a category. It is a subcategory of this one which is of interest to the algebraic geometer.

Namely, given a ring A, its spectrum $X=\operatorname{spec}(A)$ is the topological space (T_0 but not T_1) whose points are the prime ideals of A with Zariski topology (the set of primes containing a given ideal is closed). One views X as a ringed space, the sheaf being that of the local rings $A_\mathfrak{p}$. It is thus a local ringed space, called an affine scheme. A prescheme is a local ringed space (X, O_X) such that every point admits an open neighborhood U such that $(U, O_X|U)$ is isomorphic to an affine scheme. The preschemes form a category, the morphisms being the local morphisms.

To simplify the notation, one sometimes omits the structure sheaf O_X and the map ψ, just writing for instance $\phi: X\rightarrow Y$ to indicate a morphism in the category of preschemes.

Let Γ be the functor "section". For each open subset U of X, ΓU is the ring of sections of O_X over U. Given a morphism $\phi: X\rightarrow Y$, we have a homomorphism $\Gamma(\phi): \Gamma Y\rightarrow\Gamma X$. The converse is true for affine schemes, and in fact affine schemes Y are characterized among preschemes by the fact that for each prescheme X the map $\phi\rightarrow\Gamma(\phi)$ of $\operatorname{Mor}(X, Y)$ into $\operatorname{Hom}(\Gamma Y, \Gamma X)$ is an isomorphism. (One could actually let X range over local ringed spaces.) Furthermore, if $Y=\operatorname{spec}(A)$, then ΓY is naturally isomorphic to A.

The other main result of Chapter I is then given: It is the proof that products exist in the category of preschemes. Let us recall some terminology in abstract categories. Let C be a category, and S an object in C. We denote by C_S the category of objects over S, i.e. pairs (X, f) where X is in C and f is a morphism $f: X\rightarrow S$ in C, called the structural morphism. Given two objects $f: X\rightarrow S$ and $g: Y\rightarrow S$ in C_S, a morphism ϕ in C_S is a morphism $\phi: X\rightarrow Y$ in C which is such that the diagram

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

is commutative.

In the category of preschemes, the object S plays the role of a ground object (ground field, ground ring, ground anything you want vastly generalized, parameter object, etc.).

A product of two objects (X, f) and (Y, g) over S consists of an object (written $X\times_S Y$) and two morphisms

$$\phi: X \times_S Y \to X$$
$$\psi: X \times_S Y \to Y$$

making the following diagram commutative and satisfying the obvious universal mapping property for such pairs of maps:

$$\begin{tikzcd} & X \times_S Y \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

It is uniquely determined, up to a unique isomorphism.

If A, B, R are three rings, and A, B are algebras over R, then the product of the two affine schemes spec(A) and spec(B) over spec(R) is spec$(A \otimes_R B)$, the morphisms involved being the obvious ones. This is practically immediate from the definitions, and the existence proof in the general case is carried out by gluing local pieces together.

One can consider the product non-symmetrically. Viewing S as ground object, let $S' = Y$ be viewed as an extension of it. Then $X\times_S S'$ (sometimes written $X^{S'}$) may be viewed as an object over S', called the pull back of X by the morphism $g: S' \to S$. This pull back involves as a special case the extension of ground field or ring, and also reduction mod p, or the process of taking a fiber. For instance, if $S = \mathrm{spec}(Z)$ (Z the integers), then for each prime p, we have a morphism

$$\mathrm{spec}(Z/pZ) \to \mathrm{spec}(Z)$$

and thus for each prescheme X over Z, we get its fiber over Z/pZ, namely $X\times_Z \mathrm{spec}(Z/pZ)$.

Having constructed products, one gets a diagonal morphism

$$X \to X \times X$$

(the product without subscript being always over spec(Z)). One says that X is a scheme if this morphism is closed (obvious definition).

Most of the rest of Chapter I is devoted to defining certain classes of morphisms in the category of preschemes (immersions, closed immersions, local immersions, morphisms of finite type, proper morphisms, separated morphisms, etc. and in subsequent chapters affine morphisms, projective morphisms, flat morphisms, unramified morphisms, simple morphisms, ad lib.) and of proving standard properties concerning the composition and products of such special classes of morphisms. Namely, given a category C, let us say that a subclass C' of morphisms of C is distinguished if it has the following properties:

(i) If f, g are in C' and can be composed, so is fg.

(ii) If $f: X \to S$ is in C' and $g: Y \to S$ is in C, then the pull back of f by g is in C'.

(iii) If both f and g are in C', so is $f \times_S g$.

(iv) If f and g can be composed, g is in C' and gf is in C', then f is in C'.

The general rule is that all particular types of morphisms defined in Chapter I (and subsequently) will form a distinguished subclass, except possibly under certain conditions of finiteness and separation. There is no point in going into the specific details here. We wish merely to indicate the way the system works.

Chapter I concludes with an extended discussion of quasi-coherent sheaves, and formal schemes, those arising essentially from completions of topological rings, and playing an important role in local analytic (algebraic) questions. They are not used until Chapter III, which will include Zariski's theory of holomorphic functions and the connectedness theorem, and the reader may skip that part until he needs it.

One more notion appears in Chapter I, worthy of notice for the implications it has concerning the point of view of the work. Again it is best to describe it in an abstract category C. Let A be a fixed object in C and let X vary in C. Then

$$F_A: X \to \mathrm{Mor}(X, A)$$

is a (contravariant) functor from C into the category of sets, denoted Ens. We may also denote $\mathrm{Mor}(X, A)$ by $A(X)$ and in our category of preschemes, we think of it as giving the set of points of A in X. (To justify this, think of A as an affine variety V over a field k, and let Γ be its finitely generated algebra of functions over k. Let K range over fields containing k. Then points of V in K are in bijective correspondence with homomorphism of Γ into K, i.e. morphisms of $\mathrm{spec}(K)$ into $\mathrm{spec}(\Gamma)$. Here, $\mathrm{spec}(K)$ consists of one point, and the local ring above it is just K itself.)

Given a functor $F: C \to \mathrm{Ens}$ of C into the category of sets, Grothendieck calls F representable if it is isomorphic to a functor of type F_A. (The functors of one category into another form themselves a category, the morphisms being the obvious ones.) It is then immediate that the object A is uniquely determined, up to a unique isomorphism.

Observe that the definition of products has been made in accordance with the representation functor, i.e. to satisfy the formula

$$(X \times_S Y)(T) \approx X(T) \times_{S(T)} Y(T)$$

for all objects T, the fiber product on the right being the usual one in the theory of sets (pairs of points projecting on the same point in $S(T)$).

This notion of representable functor allows one to transport to any category standard notions like group, ring, etc. For instance, an object G is called a group object if one is given two morphisms $G\times G\to G$ (composition) and $G\to G$ (inverse) such that the representation functor into the category of sets defines a group structure on the set $G(X)$ for each X. (We have assumed finite products exist, but a rephrasing would do away with this.)

It is one of the most basic ones of mathematics. To give an example from topology: On the category of CW complexes, the functor H^n_π is representable by $K(\pi, n)$. Or on the category of reasonable topological spaces, the functor K (classes of vector bundles) is also representable by the classifying space.

In algebraic geometry, Grothendieck reformulates certain classical problems in terms of the representation of functors, for instance the problem of constructing Picard schemes. Given X over S, the Picard functor consists in associating to each T over S the divisor classes of X which are rational over T. (This can of course be made precise.) The Picard scheme, if it exists, represents this functor. Grothendieck has recently obtained a fairly general condition on functors in the category of schemes under which he can prove that a functor is representable. This point of view marks a complete discontinuity with those preceding it and in a certain sense, is the first essentially new approach having entered algebraic geometry since the Italian school.

A theorem is not true any more because one can draw a picture, it is true because it is functorial.

To conclude this review, I must make a remark intended to emphasize a point which might otherwise lead to misunderstanding. Some may ask: If Algebraic Geometry really consists of (at least) 13 Chapters, 2,000 pages, all of commutative algebra, then why not just give up?

The answer is obvious. On the one hand, to deal with special topics which may be of particular interest only portions of the whole work are necessary, and shortcuts can be taken to arrive faster to specific goals. Thus one may expect a period of coexistence between Weil's

Foundations and *Elements*. Only history will tell if one buries the other. Projective methods, which have for some geometers a particular attraction of their own, and which are of primary importance in some aspects of geometry, for instance the theory of heights, are of necessity relegated to the background in the local viewpoint of *Elements*, but again may be taken as starting point given a prejudicial approach to certain questions.

But even more important, theorems and conjectures still get discovered and tested on special examples, for instance elliptic curves or cubic forms over the rational numbers. And to handle these, the mathematician needs no great machinery, just elbow grease and imagination to uncover their secrets. Thus as in the past, there is enough stuff lying around to fit everyone's taste. Those whose taste allows them to swallow the *Elements*, however, will be richly rewarded.

S. LANG

Foundations of Modern Analysis. By J. Dieudonné. New York, Academic Press, 1960. 14+361 pp. $8.50.

The purpose of this book is to provide the necessary elementary background for all branches of modern mathematics involving Analysis, and to train the students in the use of the axiomatic method. It emphasizes conceptual rather than computational aspects. Besides pointing out the economy of thought and notation which results from a general treatment, the author expresses his opinion that the students of today must, as soon as possible, get a thorough training in this abstract and axiomatic way of thinking if they are ever to understand what is currently going on in mathematical research. The students should build up this "intuition of the abstract", which is so essential in the mind of a modern mathematician. The angle from which the content of this volume is considered is different from the ones in traditional texts of the same level because the author does not just imitate the spirit of his predecessors but instead has a more independent pedagogical attitude. This book takes the students on a tour of some basic results, among them the Tietze-Urysohn extension theorem, the Stone-Weierstrass approximation theorem, the Ascoli compactness theorem, the Jordan curve theorem and the F. Riesz perturbation theory. These are some of the hills in the scenery which are surrounded by nice valleys connecting them. This course, to be taught during a single academic year, is *elementary* in the sense that it is intended for *first year* graduate students or exceptionally advanced undergraduates. Naturally, students must have a good working knowledge of classical Calculus and of elementary Linear Algebra before reading this volume. The book includes a good list of problems, some of them particularly interesting and unusual for a textbook. Specific references to the books of Ahlfors, Bourbaki, Coddington-Levinson, Halmos, Jacobson, Kelley, Loomis and Taylor are included to assist the students in completing their knowledge.

Chapter I (Elements of set theory) treats the indispensable minimum about sets, Boolean algebra, product sets, mappings and denumerable sets. The author does not try to put set theory on an axiomatic basis. He remarks that one very seldom needs more than elementary properties in the applications of set theory to Analysis. The author states the axiom of choice neatly and makes no noise about it. He says that it can sometimes be shown that a theorem proved with the help of that axiom can actually be proved without it. However he never goes into such questions, which properly belong to Logic.

Chapter II (Real numbers) derives the properties of real numbers from a certain number of statements taken as axioms. The real numbers system is presented as an Archimedean ordered field satisfying the nested intervals condition. These axioms can, of course, be proved to be consequences of the axioms of the natural integers together with parts of set theory through the Dedekind or Cantor procedures. Although such proofs have great logical interest, they have no bearing whatsoever on Analysis and teachers should not burden students with them in trying to transmit the spirit of mathematical rigor. This is the right attitude shared by this text.

Chapter III (Metric spaces) constitutes the core of the book, as there is developed in it the geometric language in which we now express the results of Analysis and which has made it possible to reach full generality, besides occasionally supplying the simplest and most perspicuous proofs. As the author says, after some experience the student should be able to acquire the conviction that, with proper safeguards, his own geometric intuition is an extremely reliable guide and that it would be a real pity to limit it to ordinary three dimensional space. This chapter deals in a standard way with continuity, completeness, compactness and connectedness. The completion procedure of metric spaces is not mentioned.

Chapter IV (Additional properties of the real line) includes some elementary properties of the real number system, plus the Tietze-Urysohn extension theorem, which is proved through a known explicit formula peculiar to the metric case.

Chapter V (Normed spaces) and *Chapter* VI (Hilbert spaces) present the elementary geometrical aspects of Banach and Hilbert spaces and also discuss convergent series. Propositions on Banach spaces linked to the notion of Baire category and duality theory are not touched upon. Unwarned readers may find the author a little ungenerous concerning the amount of material in Chapter VI, which looks surprisingly short as compared to what one would expect from the warm praise of Hilbert spaces in the text.

Chapter VII (Spaces of continuous functions), after a few indispensable preliminary considerations, presents in a neat and direct form two of the basic tools of Analysis, namely the Stone-Weierstrass theorem and its application to polynomial and trigonometric approximation and the Ascoli compactness theorem in continuous functions spaces. This is a short and elegant chapter, which presents in a tidy form fundamental material not yet standard in elementary textbooks.

Chapter VIII (Differential Calculus) is beautifully written. The subject matter of the chapter is nothing else but the elementary theorems of Calculus, presented in a manner and generality not yet the vogue in textbooks of comparable level. The author is a partisan of an intrinsic formulation and a geometric outlook on Analysis through use of Banach spaces. Aside from several applications of such a general Calculus, one of the sound motivations for this intrinsic viewpoint is the idea of calculus on a manifold which no young mathematician of nowadays can ignore any longer. The author advises the readers in a fatherly way to assume all vector spaces to be finite dimensional if that gives them an additional feeling of security, but he also stimulates the students to greater courage by adding that this assumption will not make the proofs shorter or simpler. By sticking to the fundamental idea of Calculus, namely the *local* approximation of functions by *linear* functions, successive derivatives $f^p(x_0)$ at a point $x_0 \in A$ of a mapping f of an open subset A of a Banach space E into a Banach space F, are defined to be in the Banach space $\mathcal{L}_p(E; F)$ of all continuous p-linear mappings of $E^p = E \times \cdots \times E$ (p times) into F. The basic rules of Calculus are proved in this geometric setting and reproduce, of course, classical rules when $E = R^n$ and $F = R^m$ are the spaces of n and m variables. The all-important mean value theorem is proved for vector valued functions in the weak form of an inequality, which corresponds to $|f(b) - f(a)| \leq |b - a| \cdot \sup_{a \leq x \leq b} |f'(x)|$ rather than to the more precise classical form expressed as an equality. For most purposes, indeed, as the author points out, all one needs to know is the inequality formulation. The primitive and integral for functions of a real variable are not deduced from the general theory of Lebesgue integration, which has won a definitive place in Mathematics, nor from Riemann integration, which seems to have already seen its golden period and may become an antiquary item, but only for vector valued functions of real variables with discontinuities of first kind (in an awkward classical terminology), or *regulated* functions according to the author's neologism, that is a function having a limit on the right and on the left at each point. The plausibility of this choice is that integration is easily and intuitively defined for step functions and that a mapping f of a compact interval of the real line into a Banach space is regulated if and only if f is the limit of a uniformly convergent sequence of step functions, which allows one to extend integration by uniform continuity. Since the powerful tools of Lebesgue integration are not needed in a number of important questions, it is perfectly feasible to limit the integration process to a category of functions containing the continuous ones and large enough for elementary purposes. This is what the author does by stopping at regulated functions and so going only halfway to Riemann integration.

Chapter IX (Analytic functions) emphasizes only the general facts for analytic functions of a *finite* number of variables with values in Banach spaces. The cases of real and complex variables are discussed simultaneously, as far as this can be done. The presentation goes up to the Cauchy integral theorem in its usual form. Results based on the Weierstrass preparation theorem are not discussed, so that we have here the theory of analytic functions of several variables only in the elementary sense. An *Appendix to Chapter* IX (Application of analytic functions to plane Topology by Eilenberg's method) is one of the pearls in this text. An irreducible minimum concerning indexes, homotopies and essential mappings leads to elegant proofs of Janiszewski's separation theorem and the Jordan curve theorem. Some students and even mature mathematicians know the statement of the Jordan theorem but have never seen its proof. They even understand that false proofs were given by distinguished mathematicians, including Jordan himself. It is therefore welcome to have the accessible and neat proof in this elementary text, besides those already available.

Chapter X (Existence theorems) deals with procedures linked to the notion of completeness and the method of successive approximations in establishing stability theorems for local homeomorphisms under "slight" perturbations and fixed point theorems. Only the most elementary results of this type are exposed. The subject matter of this chapter has become a classical and fashionable way of introducing students to Functional Analysis, as it requires only a few abstract notions in establishing tangible classical results in a unified way. This chapter deals with the implicit function theorem for functions between Banach spaces, the Cauchy existence theorem for ordinary differential equations in vector valued functions and the Frobenius theorem on the complete integrability of total differential equations between Banach spaces.

Chapter XI (Elementary spectral theory) is the gist of the course, not only because it provides an easy approach to a powerful method of Analysis, namely Spectral Theory, but also because it draws practically on every preceding chapter, showing the students that those abstract techniques were not purposeless generalizations. The flavor of this chapter is that, following Fredholm, compact operators can be viewed as "slight" perturbations of general continuous operators, provided one considers as "negligible" what happens in finite dimensional spaces. After a few elementary properties of spectra of continuous operators, the theory of F. Riesz concerning compact perturbations of an identity operator is developed. Since this theory, for topological vector spaces, has found new geometrical applications other than those devised initially, it is nice to have it presented almost at the start of this chapter with few prerequisites. Compact operators in Hilbert spaces, the Fredholm integral equation and the Sturm-Liouville problem are the next goals of this pretty chapter.

The book is very up to date in terminology, taste and fashion. If a general program of graduate study for mathematicians is to be considered, it should be such that students are expected to get familiar with the content of this volume, whatever their future field of specialization may be. Many opinions it contains are stated in an incisive way, well-known to people personally acquainted with the author, in an attempt to eliminate some vicious attitudes repeated over and over again in traditional texts. This is a most valuable elementary book written by a distinguished mathematician which undoubtedly will help to attract fresh talent into Mathematics. In west Europe, in Japan and in the United States, it is not yet as common to have genuinely good elementary texts written by outstanding mathematicians as it is nowadays in Russia, where, in spite of printing costs, inexpensive editions make such books accessible to the pocket of almost every student.

LEOPOLDO NACHBIN

THE ANNUAL MEETING IN WASHINGTON

The sixty-seventh annual meeting of the American Mathematical Society was held at the Hotel Willard in Washington, D. C., on January 23–26, 1961. During the same week, with headquarters at the same hotel, there were meetings of the Association for Symbolic Logic, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics. The attendance of 1,527, including 1,203 members of the Society is the largest in the records of this Society.

The thirty-fourth Josiah Willard Gibbs Lecture was presented by Professor J. J. Stoker of New York University. The high quality of the series was beautifully upheld. The title of the lecture was *Problems in nonlinear elasticity*. President Montgomery presided at the session, which took place at 8:00 P.M. on Tuesday, January 24, in the Grand Ballroom.

By invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings, Professor Lars Hörmander of the University of Stockholm and the Institute for Advanced Study addressed the Society in the Grand Ballroom at 2:00 P.M. on Monday, January 23. The title of his address was *On the range of differential operators*. Dr. Vlastamil Pták introduced the speaker.

By invitation of the same committee, Professor Helmut Wielandt of the University of Tübingen and the California Institute of Technology gave an address in the Grand Ballroom at 2:00 P.M. on Wednesday, January 25. His subject was *On the structure of finite groups*. The speaker was introduced by Professor Marshall Hall, Jr.

There were thirty-two sessions for contributed papers, at which a total of two hundred thirty-four papers were presented to the Society. The number of papers was somewhat larger than the number presented at the previous annual meeting and established a new record for the number of papers presented. The Society acknowledges with thanks the service of the following people in presiding over the sessions for contributed papers: Dr. Norman Bazley, Professors R. H. Bing, R. H. Cameron, V. F. Cowling, J. H. Curtiss, J. B. Diaz, Dr. R. D. Driver, Professors Carl Faith, D. J. Foulis, Murray Gerstenhaber, O. G. Harrold, Jr., Mr. Michael Goldberg, Professors Emil Grosswald, Erik Hemmingsen, J. E. Houle, Jr., S. B. Jackson, V. L. Klee, Jr., Jean E. LeBel, Dr. Benjamin Lepson, Professors L. F. McAuley, Josephine Mitchell, David Nelson, B. E. Rhoades, Walter Rudin, Anne E. Scheerer, Dr. Daniel Shanks, Dr. Oved Shisha, Professors M. F. Smiley, Andrew Sobczyk, Domina E. Spencer, Choy-Tak Taam, and A. C. Zaanen.

As the meetings of this Society grow, it becomes increasingly necessary to use mechanical aids to the ear and eye in order that a large audience may follow all the details of a lecture. Of special note at this meeting was the fact that, with the cooperation of the manufacturer, the projection machine called the Vu-Graph was available at every session. In addition, instruction was available about the use of the machine and materials could be procured for advance preparation of transparencies for projection during lectures. In the hands of a speaker who takes the pains to make careful preparation, the Vu-Graph appears to be quite successful as an aid to the eye of the members of the audience.

Abstracts of the papers presented at the meeting appear in the Notices of the American Mathematical Society for December (issue no. 50) and succeeding issues. There were no papers presented by title except for the isolated instances of contributors who were unavoidably prevented from presenting in person a paper which had already been scheduled. This was the first meeting affected by the recently instituted plan of handling contributed papers for presentation by title through supplementary programs which are detached from any meeting of the Society.

The Annual Business Meeting of the Society was held in the Grand Ballroom at 2:00 P.M. on Thursday, January 26, 1961. Vice President Bohnenblust presided.

Appreciation was expressed to Professors John W. Brace and M. W. Oliphant for their contributions to the arrangements for the meetings.

Dr. W. Homer Turner, Executive Director of the United States Steel Foundation, made a short address to the business meeting and presented the Society with a check for five thousand dollars to assist in studying modes of communication between modern developments in mathematics and graduate students, and between the industrial mathematician and the academic mathematician.

The Secretary reported briefly on the affairs of the Society.

The Trustees of the Society met on the morning of Tuesday, January 26.

The Council of the Society met on the afternoon of Wednesday, January 25. After an intermission for dinner the meeting continued through the evening.

At the Council meeting, the Secretary announced the election of the following one hundred twenty-two persons to ordinary membership in the Society:

Mr. Oliver G. Aberth, Swarthmore College;
Mr. Achdat, Bandung Institute of Technology, Bandung, Java, Indonesia;
Mr. David J. Allen, Donell Farm, Bennington, Vermont;
Dr. Donald E. Amos, Sandia Corporation, Albuquerque, New Mexico;
Miss Kay A. Anderson, Analyst, Computer Sciences Corp., Inglewood, California;
Professor Kiyoshi Aoki, Niigata University, Niigata, Japan;
Mr. Lucio Artiaga, University of Saskatchewan;
Mr. George W. Batten, Jr., William Marsh Rice University;
Mr. Joseph Battle, University of Michigan;
Mr. James E. Benson, Fairleigh Dickinson University;
Mr. Richard W. Benson, Research Math., Pikewood Corp., Albuquerque, New Mexico;
Mr. Triloki N. Bhargava, Michigan State University;
Mr. Daniel G. Bobrow, Massachusetts Institute of Technology;
Mr. James R. Bower, University of Michigan;
Mr. Edwin H. Brackett, International Business Machines Corporation, Bethesda, Maryland;
Mrs. Olive S. Bowman, Bridgewater College;
Professor James R. Brown, University of Massachusetts;
Mr. Hernan R. Bravo Flores, Institute de Fisica y Mathmaticas, Chile;
Mr. Edward S. Brown, Jr., Defense Atomic Support Agency, Albuquerque, New Mexico;
Mr. Richard A. Byerly, The Association for Bank Audit Control and Operation, Clarendon Hills, Illinois;
Mr. Gaylord A. Capes, Westinghouse Electric Corp., Baltimore, Maryland;
Dr. Roderick G. Chisholm, St. Mary's College;
Mr. Donald H. Clanton, Oak Ridge National Laboratory, Oak Ridge, Tennessee;
Mr. William E. Christilles, St. Mary's College;
Mr. Richard L. Cline, International Business Machines Corp., New York, New York;
Mr. David B. Coghlan, Foote Mineral Company, Berwyn, Pennsylvania;
Sister Conrad, Central Catholic High School, Ft. Wayne, Indiana;
Mr. George T. Crocker, Auburn University;
Professor Ubiratan D'Ambrosio, University of São Paulo;
Professor Ludwig W. Danzer, University of Washington;
Mr. Gary A. Davis, Mt. Carmel College, Niagara Falls, Ontario, Canada;
Mr. James R. Dean, Technical Operations, Inc., Fort Monroe, Virginia;
Mr. Donald F. Dempsey, International Business Machines Corp., Dearborn, Michigan;
Mr. Louis E. De Noya, Oklahoma State University;
Professor Nicolae Dinculeanu, University of Bucarest, Bucarest, Roumania;
Dr. Andrew G. F. Dingwall, Radio Corporation of America, Harrison, New Jersey;
Miss Diane K. Downie, University of Idaho;
Mr. John R. Durbin, University of Kansas;
Dr. David B. A. Epstein, Princeton University;
Dr. Manus R. Foster, Socony Mobil Oil Co., Dallas, Texas;
Mr. Bernard L. Freese, Royal McBee Corp., Chicago, Illinois;
Mr. Robert B. Gardner, University of California;

Mr. Francis J. Garvis, System Development Corp., Santa Monica, California;
Mr. Thomas M. Gill, Bethel College;
Mr. Samuel Gorenstein, System Development Corp., Paramus, New Jersey;
Professor Svend T. Gormsen, Virginia Polytechnic Institute;
Mr. George C. Graff, University of Illinois;
Mr. Frederick P. Greenleaf, Yale University;
Dr. George J. Habetler, General Electric Co., Schenectady, New York;
Dr. Asghar Hameed, Government College of Engineering and Technology, Pakistan;
Dr. Eldon R. Hansen, Lockheed Aircraft, Palo Alto, California;
Mr. Warren F. Haverkamp, Columbia Wax Co., Glendale, California;
Mr. Earl W. Hessee, Lockheed Aircraft Corp., Dawsonville, Georgia;
Mr. Kenneth L. Hillam, University of Colorado;
Mr. William W. Hokman, Virginia Polytechnic Institute;
Professor Kinya Honda, St. Paul's University, Tokyo, Japan;
Mr. Donald G. Hook, University of California;
Professor Paul H. Hutcheson, Middle Tennessee State College;
Mr. Douglas H. Hutchinson, Union Carbide Consumer Products Co., Cleveland, Ohio;
Professor John M. Irwin, New Mexico State University;
Mr. Robert C. Irwin, MITRE Corp., Bedford, Massachusetts;
Miss Joanne M. Jasper, Anna Maria College;
Mr. Robert E. D. Jones, Iowa State University;
Mr. William B. Jones, National Bureau of Standards, Boulder, Colorado;
Dr. Frank C. Karal, Jr., New York University;
Mr. Seymour Kass, University of Chicago;
Lt. Jerald C. Kindred, Air Force Department, Ft. Lawton, Washington;
Mr. Jerry P. King, University of Kentucky;
Mr. Kenneth R. Klopf, Shell Oil Co., Midland, Texas;
Mr. Bengt J. Kredell, ASEA, Vasteras, Sweden;
Mr. Isaac C. Lail, Army Department, Washington, D. C.;
Professor Lawrence H. N. Lee, University of Notre Dame;
Mr. Kenneth D. Lerche, Lehigh University;
Mr. Stanley M. Lukawecki, Auburn University;
Professor Gustave H. Lundberg, Vanderbilt University;
Reverend John J. MacDonnell, College of the Holy Cross;
Mr. Walter T. Mara, Monterey Peninsula College;
Dr. Eugene H. Nicholson, St. Louis, Missouri;
Mr. Torsten Norvig, University of Massachusetts;
Dr. Ernest L. Osborne, Washington, D. C.;
Mr. Jack L. Owens, Mene Grande Oil Co., Apartado 45, Barcelona, Venezuela;
Mrs. Frankie B. Patterson, Southern University;
Mr. Donald M. Peterson, Sr., Convair Corp., Ft. Worth, Texas;
Mr. Michael R. Pew, Electra Mfg., Independence, Kansas;
Mr. John A. Pfaltzgraff, University of Kentucky;
Mr. John T. Porter, Jr., Canateson Salvage, Moody, Texas;
Mf. Alfred G. Quade, Pure Oil Co., Chicago, Illinois;
Mr. Louis V. Quintas, City College, New York, New York;
Mr. Marlon C. Rayburn, Jr., Earlham College;
Mr. Clyde D. Rinker, Bendix Corp., Kansas City, Missouri;
Mr. Edwin H. Rogers, Carnegie Institute of Technology;
Dr. Bernard W. Roos, General Dynamics Corp., San Diego 12, California;

Professor Paul T. Rygg, Montana State University;
Mr. William L. Salvatore, Classical High School, Providence, Rhode Island;
Miss Mary F. Saunders, Sanders Association, Inc., Nashua, New Hampshire;
Mr. Richard L. Schauer, University of Wisconsin;
Mr. Alvin L. Schreiber, Human Sciences Research, Inc., Arlington, Virginia;
Dr. Lorraine Schwartz, University of British Columbia, Vancouver;
Professor Robert E. Seal, Illinois Institute of Technology;
Mr. Freeman S. Sharp, Hyattsville, Maryland;
Dr. Isadore Silberman, Raytheon Corporation, Bedford, Massachusetts;
Mr. Michael S. Skaff, University of Illinois;
Mr. William T. Sledd, University of Kentucky;
Dr. Alan R. Smith, International Nickel Co. of Canada, Thompson, Manitoba;
Mrs. Dorothy P. Smith, New Mexico Highland University;
Mr. Robert P. Smith, Navy Department, Washington, D. C.;
Mr. Robert E. Spivack, University of South Carolina;
Brother Joseph W. Stander, University of Dayton:
Mr. Jeremy J. Stone, Stanford Research Institute;
Professor Charles F. Taylor, Maryville College;
Dr. Sean J. Tobin, University College, Galway, Ireland;
Professor Terry Triffet, Michigan State University;
Mr. Verlyn R. Unruh, System Development Corp., Santa Monica, California;
Professor Gerard J. Van Der Maas, University of Ottawa, Ottawa, Ontario;
Mr. John S. Warren, Boston Edison Co., Boston, Massachusetts;
Miss Martha F. Watson, University of Kentucky;
Professor James R. Webb, Louisiana State University;
Professor Arthur D. Wirshup, California State Polytechnic College;
Dr. N. Donald Ylvisaker, Columbia University;
Mr. Raymond A. Zachary, Jr., Texas Instruments Inc., Dallas, Texas;
Dr. James P. Zietlow, New Mexico Highland College.

It was reported that two hundred and ninety-six persons were elected to membership on nomination of institutional members as indicated:

University of Alberta:John W. Moon.
Andrews University: Mr. Theodore R. Hatcher.
Arizona State University: Professor Robert W. Sanders.
Auburn University: Mr. Porter G. Webster.
University of British Columbia: Miss Marguerite E. Barrett, Mr. Jay L. Delkin, Mr. Alan R. Dobell, Mr. Gene B. Gale, Mr. William T. Iwata, Mr. Robert L. Johnston, Mr. Richard Lee, Mr. Donald J. Mallory, Mr. Frank C. May, Mr. Richard C. Willmott.
Brown University: Mr. Frederick J. Almgren, Jr., Mr. Paul Dormont, Mr. Joseph B. Geiser, Mr. Morton E. Gurtin, Dr. Robin J. Knops, Professor Allen C. Pipkin, Dr. Tryfan G. Rogers, Mr. Gordon B. Small, Jr., Mr. William F. Tyndall, Mr. Eric Varley, Mr. Michael Voichick.
California Institute of Technology: Mr. Stephen A. Andrea, Mr. Richard E. Balsam, Mr. Fletcher I. Gross, Mr. Alfred W. Hales, Mr. Donald E. Knuth, Mr. Louis A. Lopes, Jr., Mr. Jack W. Macki, Mr. Stanley A. Sawyer.
University of California, Berkeley: Mr. Bruce A. Bloomfield, Mr. Fraser A. Bonnell, Mr. Benson S. Brown, Miss Carole J. Colebob, Mr. Ernest T. Fickas, Mr. Haim

Gaifman, Mr. Ronald L. Graham, Mr. Michel Jean, Mr. William L. Kent, Mr. Charles P. Luehr, Mr. George H. Orland, Mr. George S. Rinehart, Mr. Galen L. Seever, Mr. William H. Sills, Mr. Eleftherios C. Zachmanoglou.

University of California, Los Angeles: Mr. Lawrence P. Belluce, Mr. Stuart E. Black, Mr. Stanley P. Franklin, Mr. William M. Lambert, Jr., Mr. Ralph H. Wessner.

Case Institute of Technology: Mr. Charles G. Cullen.

Catholic University of America: Mr. Gerald R. Andersen, Rev. John C. Friedell.

University of Chicago: Mr. Robert B. Brown, Mr. Leif Kristensen, Mr. Tzee C. Kuo, Mr. Yung-Yung Lu, Miss Therese E. Raczynski, Mr. Mitchell H. Taibleson.

University of Colorado: Mr. John D. De Pree, Mr. David A. Shotwell.

Cornell University: Dr. Zbigniew Ciesielski, Dr. Caspar R. Curjel, Mr. Harold G. Diamond, Mr. Paul S. Green, Mr. Clifford T. Ireland, Mr. Alan McConnell, Mr. Stanley E. Mamangakis, Mr. Alfred B. Manaster, Mr. Irwin S. Pressman, Mr. John S. Rose, Mr. Chia-Hui Shih, Mr. Benjamin T. Smith, Dr. Samuel J. Taylor.

Duke University: Mr. David R. Anderson, Mr. Warren S. Edelstein, Mr. Dick L. George, Mr. David R. Hayes, Mr. Robert M. McConnel.

University of Florida: Mr. Billy R. Hare.

University of Georgia: Mr. Julio R. Bastida, Mr. Curtis P. Bell, Mr. Brittian J. Williams.

Harvard University: Mr. Edward B. Curtis, Mr. Jerry L. Fields, Mr. Bernard R. Kripke, Mr. Satish D. Shirali.

Illinois Institute of Technology: Mr. Eugene L. Allgower.

University of Illinois: Mr. Steven F. Bauman, Mr. Clinton R. Foulk, Mr. James J.. Gillian, Mr. Charles F. Koch, Mr. Charles G. Krueger, Mr. Gary K. Leaf, Mr. Eizo Nishiura, Mr. Surendra N. Patnaik, Dr. Chivukula R. Rao, Mr. Keith A. Rowe.

Institute for Advanced Study: Dr. James A. Green, Professor Tatsuo Homma, Professor Heinz Huber, Professor Jan W. Jaworowski, Dr. Mieo Nishi, Mr. Mikio Sato, Dr. Charles T. C. Wall.

Institute for Defense Analyses: Mr. Gerald J. Mitchell.

State University of Iowa: Mr. Orabi H. Alzoobaee, Mr. Norman Y. Luther, Mr. Joseph M. Martin, Mr. Donald V. Meyer.

Johns Hopkins University: Mr. George E. Lindamood, Mr. Peter H. Lord, Mr. Charles C. Pugh, Mr. Allan J. Silberger, Mr. Morris L. Thrower, Mr. J. Thomas Warfield, Miss Bernice Weinstein.

University of Kansas: Mr. Terrence J. Brown, Mr. William T. Covert, Mr. Eberhard G. P. Gerlach, Mr. J. Peter Johnson, Mr. Martin T. Lang, Mr. Paul W. Liebnitz, Mr. William D. McIntosh, Dr. Andrew Page, Mr. Raymond E. Pippert.

Lehigh University: Mr. Gerard E. Cozzolino, Mr. David K-s Hsieh.

McMaster University: Mr. Howard L. Jackson, Professor Derek J. Kenworthy.

University of Maryland: Mr. George R. Desi, Mr. Donald H. Flanders, Mr. Robert J. Gauntt, Mr. Svetozar Kurepa, Mr. Richard J. Weinacht, Professor Marvin Zelen.

Massachusetts Institute of Technology: Mr. Paul W. Abrahams, Mr. Howard E. Conner, Mr. Ramesh A. Gangolli, Mr. Joseph Hershenov, Mr. Louis Hodes, Mr. Gerald M. Leibowitz, Mr. David C. Luckham, Mr. Richard M. Moroney, Jr., Mr. William E. Ritter, Mr. Gabriel Stolzenberg, Mr. Norman R. Wagner, Mr. Israel J. Weinberg.

University of Miami: Mr. Charles R. Fitzpatrick, Mr. Jules B. Kaplan.

Michigan State University: Mr. John W. Baker, Mr. Mickey W. Dargitz, Mr. Donald L. Fisk, Mr. Jerome X. Goldschmidt, Mr. Robert L. Hemminger, Mr. David C. Kay.

University of Michigan: Mr. George H. Andrews, Professor Robert J. Bridgman, Mr. John A. A. Kelingos, Mr. Patrick J. Ledden, Mr. Gerald E. Meike, Mr. Donald E. Sarason, Mr. George R. Sell, Professor Khyson Swong, Mr. Charles A. Trauth, Jr., Mr. Bertram J. Walsh.

University of Minnesota: Mr. Patrick R. Ahern, Mr. Jay P. Fillmore, Mr. C. J. Norman Fritz, Mr. Wayne W. Schmaedeke, Mr. James W. Yackel.

University of Missouri: Mr. William A. Kirk, Mr. Eugene F. Steiner, Mr. Paul E. Waltman.

University of Nebraska: Mr. Jerrold W. Bebernes.

University of New Hampshire: Mr. Robert E. O'Malley, Jr.

New Mexico State University: Mr. William E. Walden.

New York University: Mr. Yung M. Chen, Mr. Djairo G. De Figueiredo, Mr. George W. Logemann, Professor Sigeru Mizohata, Mr. Kennard W. Reed, Jr., Mr. Alan D. Solomon, Mr. George R. Stell, Mr. Robert E. L. Turner.

State University of New York: Miss Patricia L. Bihr.

University of North Carolina: Mr. Anil K. Bose, Mrs. Rebecca S. Cox, Mr. John R. Dowdle, Mr. Paul M. LeVasseur, Mr. Robert E. Spencer, Mr. Clifton T. Whyburn.

Ohio State University: Mr. Robin W. Chaney, Mr. Robert L. McFarland, Miss Joan E. Smith.

University of Oklahoma: Mr. Forrest R. Miller, Jr., Mr. David R. Proctor, Mr. Eugene E. Slaughter, Jr.

Oklahoma State University: Mr. David R. Cecil, Professor Glen A. Haddock.

University of Oregon: Mr. Robert M. Fesq, Mr. Lowell A. Hinrichs, Mr. Raymond E. Smithson, Mr. Charles L. Vanden Eynden.

Pennsylvania State University: Mr. Joseph A. Cima, Mr. Alan S. Cover, Mr. Barry F. Kramer.

Princeton University: Mr. Christopher Anagnostakis, Mr. R. Gordon Barker, Mr. Lutz Bungart, Mr. William G. Faris, Mr. John A. Hartigan, Mr. Robert C. Hartshorne, Mr. Peter J. Kahn, Mr. J. Peter May, Mr. Stephen Scheinberg, Mr. John J. Simon, Mr. Michael D. Spivak, Mr. William A. Veech, Mr. Robert Wells, Mr. Seth I. Zimmerman.

Purdue University: Mr. John R. Alexander, Jr., Mr. Robert D. Bechtel, Mr. Richard E. Hughes, Mr. Kenneth R. King, Mr. Frank A. Smith, Mr. Joel A. Smoller, Mr. John R. Sorenson, Mr. Jack R. Stodghill.

Rice University: Mr. Norman A. Shenk, II.

Rutgers, The State University: Mr. Michael W. Lodato, Mr. Chung L. Wang, Mr. Israel Zuckerman.

University of Southern California: Professor Gunter Ewald, Mr. Arthur S. Leslie, Miss Emily B. A. McCormick, Mr. Guillermo Restrepo, Professor Koichi Yamamoto.

South Dakota School of Mines and Technology: Mr. Martin J. Marsden.

Stanford University: Mrs. Patricia W. Beckman, Mr. William H. Berry, Mr. Bradley Efron, Mr. Martin Engert, Mr. Henry E. Pettis, Mr. Louis A. Fine, Mr. Jon D. Hopper, Mr. Leroy V. Junker, Mr. Franklin Lowenthal, Mr. Joseph Novello, Mr. James M. Ortega, Mr. Lawrence M. Perko, Mr. Jon E. Petersen, Mr. William H. Row, Jr., Miss Margaret E. Salmon, Mr. Dale W. Thoe, Mr. Arthur W. J. Ullman, Mr. Robert E. Wellck.

Stephen F. Austin State College: Mr. R. G. Dean.

Syracuse University: Mr. Stanley I. Mack.

University of Texas: Mr. William C. Bean, Mr. Saul I. Drobnies, Mr. Donald J. Hansen, Miss Blanche J. Monger, Mr. Douglas R. Stocks, Jr., Mr. Dale E. Walston.

University of Toronto: Mr. Edward J. Barbeau, Mr. Alan S. Deakin, Mr. Andrew J. Korsak, Mr. Donald R. Miller, Professor Kunio Murasugi.

Tulane University: Mr. Sigmund N. Hudson, Mr. Harold D. Kahn, Mr. Harry T. Mathews, Mr. David E. Penney, Mr. Walter J. Schneider.

Vassar College: Miss Sandra A. Hayes.

University of Virginia: Mr. Alvin B. Owens, Mr. Thomas W. Page, Professor George K. Williams.

Washington University: Mr. Richard A. Hunt, Mr. Arthur E. Obrock, Mr. Bobba S. Reddy.

Washington State University: Mr. William S. Eberly.

Wayne State University: Mr. John C. Cantwell, Mr. Ronald J. Knill, Mr. John O. Riedl.

College of William and Mary in Norfolk: Miss Ellen Stone.

Yale University: Mr. Laurence R. Alvarez, Mr. John D. Ferguson, Mr. John N. Frampton, Mr. Claude C. Thompson, Mr. Hoyt D. Warner.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Wiskundig Genootschap te Amsterdam: Mr. Jan R. Strooker, Professor Jacobus H. Van Lint; Australian Mathematical Society: Dr. James H. Michael; Austrian Mathematical Society: Professor Wolfgang Schmidt; Dansk Matematisk Forening: Mr. Palle F. Schmidt; Deutsche Mathematiker Vereinigung: Professor Friedemann W. Stallman, Dr. Joseph F. Weier; Société Mathématique de France: Mr. Dean M. Abadie; Indian Mathematical Society: Professor Ram Behari, Professor Phatik C. Chatterjee, Dr. Mohindar S. Cheema, Professor V. Ganapathy Iyer, Professor V. Sankriti Krishnan, Professor Bangalore S. Madhavarao, Professor Ratan S. Mishra; Mathematical Society of Japan: Professor Eizi Inaba, Professor Seizo Ito, Professor Hitoshi Iyoi, Professor Koiti Konda, Professor Tadao Kubo, Professor Katsuhiko Masuda, Professor Isamu Mogi, Professor Osamu Nagai, Professor Toshio Nonaka, Professor Yuzo Utumi, Professor Hidekazu Wada, Professor Kaneo Yamada, Mr. Yukihiro Kodama, Mr. Satio Okada; London Mathematical Society: Dr. Robin O. Gandy; Dr. Mary R. Rees; Polskie Towarzystwo Matematiyczne: Professor Marek Fisz; Suomen Matemaattinen Yhdistys: Dr. Jussi I. Vaisala; Svenska Matematikersamfundet: Mr. Sture Danielson, Mr. J. Torgny Domar, Mr. Matts R. Essen; Unione Matematica Italiana: Dr. Ferrante Pierantoni.

It was reported that J. M. Thomas represented the Society at the Fiftieth Anniversary Celebration of North Carolina College; Edward S. Hammond represented the Society at the Inauguration of Robert Edward Lee Strider II of Colby College; and Robert A. Rosenbaum represented the Society at the Silver Convocation of the University of Connecticut honoring President Albert N. Jorgensen.

The following committee appointments of the President were reported: to the *Committee on Applied Mathematics*: G. E. Forsythe and V. Bargmann; to the *Committee to Select Speakers for Summer and Annual Meetings*: Saunders MacLane; to the *Committee to Select Speakers for Eastern Meetings*: W. L. Chow; to the *Committee to Select Speakers for Western Meetings*: M. Heins; to the *Committee to Select Speakers for Far Western Meetings*: Ernst Straus; to the *Committee to Select Speakers for Southeast Meetings*: Kirk Fort; to the *Visiting Lectureship Committee*: J. L. Kelley; to the *Committee to Select Four Members of the Council to run for the Executive Committee*: Garrett Birkhoff, Edwin Hewitt, and F. Burton Jones; *as tellers for the* 1960 *election*: P. C. Curtis and R. J. Blattner; *Committee on the Cole Prize in Number Theory* to be awarded in January 1962: D. H. Lehmer, Chairman, Serge Lang and S. Chowla; to the *Joint Committee on the Doctor of Arts Degree*: E. E. Moise, Chairman, M. M. Day, Paul Halmos, and A. D. Wallace; to the *Invitations Committee for the Summer Institute on* "Applications of Functional Analysis in 1961," P. D. Lax, Chairman, R. S. Phillips and Henry Helson; to the *Committee to Nominate Officers and Members of the Council for the* 1961 *election*: E. E. Floyd, Chairman, P. R. Halmos, Edwin Hewitt, W. T. Martin and Hans Samelson; to the *Invitation Committee for a Symposium on Mathematical Problems in the Biological Sciences in April* 1961: S. M. Ulam, Chairman, A. Bartholomay, R. Bellman, J. Jacquez, T. T. Puck, and Claude Shannon.

The Secretary reported that the following have accepted invitations to deliver Hour Addresses before the Society: Gail S. Young, November, 1960, Vanderbilt University; Graham Higman, November, 1960, Northwestern University; Helmut Wielandt and Lars Hörmander at the Annual Meeting, 1961; Henry Helson, November, 1961, Santa Barbara, California; R. A. Beaumont, Far Western Meeting in Spring, 1962; Israel N. Herstein and James A. Jenkins, April, 1961, Chicago Meeting.

Upon the recommendation of the Committee on Translations, the Council recommended to the Trustees that a new publication series devoted to the translation of papers in probability and statistics be established. The Council also recommended that starting January 1, 1961, the Chinese Journal "ACTA MATHEMATICA SINICA" be translated in toto and published by the Society.

The Council, acting upon the recommendation of the Joint Committee on the Doctor of Arts Degree under the Chairmanship of Professor E. E. Moise, voted to support the action of the Board of Governors of the Mathematical Association of America in their recommendation that the degree of Doctor of Arts be established, in mathematics, at most of the universities which are qualified to grant the Ph.D.

The Council approved a recommendation of the Transactions Editorial Committee that its membership be increased to five and elected Professor Michel Loève to fill the three year term created by the Council's action.

The Council voted to elect Professor L. J. Paige as Acting Secretary of the Society for the period February 1, 1961, to September 16, 1961, in the absence of Professor John W. Green.

LOWELL J. PAIGE,
Acting Secretary
EVERETT PITCHER,
Associate Secretary
J. W. GREEN,
Secretary

THE FEBRUARY MEETING IN NEW YORK

The five hundred seventy-seventh meeting of the American Mathematical Society was held at Hunter College, New York, New York, on Saturday, February 25, 1961. About 111 persons attended, including 100 members of the Society.

By invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings, Professor John Wermer of Brown University delivered an address entitled *Uniform approximation and maximal ideal spaces.* Professor C. R. Adams presided at the session and introduced the speaker.

There were thirteen contributed papers scheduled in a morning session and an afternoon session, over which Professors Selby Robinson and James Singer presided.

EVERETT PITCHER,
Associate Secretary

RESEARCH PROBLEMS

1. Richard Bellman: *Differential equations.*

It was shown by Hermite and others that the study of the doubly-periodic solutions of a linear differential equation whose coefficients are analytic doubly-periodic functions of a complex variable is considerably simpler in many ways than the study of the periodic solutions of a linear differential equation with periodic coefficients.

One should in this way be able to obtain excellent approximations to the solution of the Mathieu equation

$$u'' + (a + b \cos 2z)u = 0$$

by considering it as a limiting form of the solution of

$$u'' + (a + b \operatorname{cn} 2z)u = 0$$

as the modulus k^2 tends to zero.

Are there doubly-periodic solutions of the inhomogeneous Van der Pol equation

$$u'' + \lambda(u^2 - 1)u' + u = a \operatorname{cn} \omega z,$$

and can these be used to furnish approximations to the solution of the equation

$$u'' + \lambda(u^2 - 1)u' + u = a \cos \omega z?$$

(Received February 2, 1961.)

2. Richard Bellman: *Asymptotic control theory.*

Consider the problem of determining the minimum of

$$J(u) = \int_0^T (u'^2 + u^2 + u^4)dt$$

over all functions $u(t)$ for which $u(0) = c$. Write $f(c, T) = \min_u J(u)$. It follows from the functional equation approach of dynamic programming that $f(c, T)$ satisfies the nonlinear partial differential equation

$$f_T = \min_v [v^2 + c^2 + c^4 + vf_c].$$

Since $f(c, T)$ is monotone increasing in T and is uniformly bounded (as we see using the trial function

$$u_0 = \frac{ce^t}{1 + e^{2T}} + \frac{ce^{2T-t}}{1 + e^{2T}},$$

the solution of the corresponding problem where the u^4 term is not present), we expect the limit function $f(c) = \lim_{T\to\infty} f(c, T)$ to satisfy the ordinary differential equation

$$0 = \min_v \left[v^2 + c^2 + c^4 + vf'(c)\right].$$

Establish this and obtain an asymptotic expansion for $f(c, T)$ and for the minimizing function u valid as $T \to \infty$. Generalize by obtaining corresponding results for the minimum of

$$J(u_1, u_2, \cdots, u_N) = \int_0^T [Q(u_1, u_2, \cdots, u_N, u_1', u_2', \cdots, u_N') + P(u_1, u_2, \cdots, u_N)]dt,$$

where Q is a positive definite quadratic form in u_i and u_i' and P is a positive polynomial of higher degree.

Results of this type are important in the modern theory of control processes. (Received February 2, 1961.)

REPORT OF THE TREASURER

The Treasurer this year again presents to the membership an abridged statement of the Society's financial position, set up in semi-informal narrative style. A copy of the complete Treasurer's Report as submitted to the Trustees and the Council will be sent to any member requesting it from the Treasurer at the Providence office. Moreover, the Treasurer will be happy to answer any questions members may wish to put to him concerning the Society's financial affairs.

The substantial increase in the General Funds of the Society ($103,280.91) is largely a reflection of certain changes in the accounting system—specifically, with respect to the valuation of equipment and the apportionment of overhead. On the same basis as last year the General Funds would have increased $23,093.58.

Returns on invested funds this year have been at the rate of 4.47% computed on book value after deduction of custodial expense. This is very slightly less than last year.

I

A Description of the Financial Position of the Society as of May 31, 1960

The Society had Cash on deposit		
In the Rhode Island Hospital Trust Company	$ 22,458.53	
In interest-bearing savings accounts	1,450.02	
In petty cash and drawing accounts in Providence and Los Angeles	1,955.00	
		$ 25,863.55
It had reserves invested until needed in Government bonds		246,846.71
There was owing to it		
By the United States Government	$ 32,870.66	
By members, subscribers and others (less allowance for doubtful accounts)	37,588.97	
		70,459.63
It had in stamps and in the postage meter		1,161.64
It had funds temporarily advanced to certain special accounts		4,713.04
And it had invested in its Headquarters Building and Office Equipment		115,738.62
Making a total of Current and Fixed Assets of		$464,783.19
The Society also held investment securities valued at		380,413.01
(The market value, May 31, 1960 was $440,281.19)		
Total Assets, therefore, were		$845,196.20

Offsetting these assets, the Society

Owed members, subscribers and vendors	$ 6,840.19	
Held funds received from various special sources to support particular projects, such as the Summer Institute, Summer Seminar, etc.	167,786.47	
Had advanced for recovery from future sales for various Society publications—Colloquium and Survey volumes, Birkhoff papers, Translation series, etc.	12,963.27	
Owed to the invested fund account	21,270.19	
And held in its General Fund the sum of	255,923.07	
Thus accounting for all the CURRENT FUNDS		$464,783.19

The Invested Funds represent the following:

(1) The Endowment Fund, largely the gift of members about thirty-five years ago	$100,000.00	
(2) The Library Proceeds Fund, derived from the sale of the Society's Library in 1950	66,000.00	
(3) The Prize Funds—Bôcher, Cole, Moore	6,575.00	
(4) The Mathematical Reviews Fund, a gift received in 1940 to make possible the establishment of the Reviews	80,000.00	
(5) Reserves established by the Trustees to protect the life memberships formerly available, and as a "hedge" against investment losses	96,286.54	
(6) Other funds, derived mainly from bequests to the Society by members, which the Trustees were either required to invest or which they have invested at their option—the income being used for the general purposes of the Society	31,551.47	
A total of INVESTED FUNDS of		380,413.01
TOTAL LIABILITIES and FUND RESERVES, therefore, were		$845,196.20

II

An Account of the Financial Transactions of the Society During the Fiscal Year 1959–1960

The Society has two types of receipts—funds for special purposes and projects, and the General Fund, from which are met the general operating expenses of the organization, including the publication of the Bulletin, the Proceedings, the Notices, and the Transactions. Income from sales of and subscriptions to these journals is placed in the General Fund, but in practice is allocated to the expenses of the journals themselves. It is so treated in the following presentation:

To meet its General obligations, the Society Received:		
From dues and contributions of individual members	$ 83,973.20	
From dues of institutional members	35,350.00	
From dues of corporate members	17,000.00	
From sales and support of scientific journals of the Society	247,334.47	
From recovery of indirect costs	48,628.89	
From investment and trusts	29,037.56	
From publication charges	2,755.00	
From meeting fees	3,050.50	
From miscellaneous sources	2,609.62	
Total General Receipts		$469,739.24
These funds were Expended		
For general administrative and meeting expenses	48,628.89	
For expenses of scientific journals of the Society	336,987.18	
In subsidies to non-Society publications	12,095.60	
For miscellaneous expenses	3,409.92	
Total General Expenses		401,121.59
Leaving an Excess of Income over Expenses of (Which was added to the General Fund)		$ 68,617.65

Scientific Journals of the Society

	Income	*Expenses*	*Deficit*
Bulletin	$ 12,231.32	$ 24,074.95	$11,843.63
Proceedings	9,350.87	36,602.83	27,251.96
Notices	12,082.50	31,536.64	19,454.14
Transactions	51,794.83	64,032.87	12,238.04
Mathematical Reviews	161,874.95	180,739.89	18,864.94
Totals	$247,334.47	$336,987.18	$89,652.71

Respectfully submitted,
Albert E. Meder, Jr.
Treasurer

December 31, 1960

RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

A NEW CLASS OF PROBABILITY LIMIT THEOREMS

BY JOHN LAMPERTI[1]

Communicated by J. L. Doob, December 30, 1960

Suppose that $\{X_n\}$ is a Markov process with states on the non-negative real axis and stationary transition probabilities. Define

$$\mu_k(x) = E[(X_{n+1} - X_n)^k \mid X_n = x], \qquad k = 1, 2, \cdots; \tag{1}$$

we assume that for each k, $\mu_k(x)$ is a bounded function of x. Assume also

$$\lim_{x\to\infty} \mu_2(x) = \beta > 0, \qquad \lim_{x\to\infty} x\mu_1(x) = \alpha > -\frac{\beta}{2}\cdot \tag{2}$$

We shall say that the process $\{X_n\}$ is *null* provided that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \Pr(X_i \leqq M) = 0 \tag{3}$$

for all finite M. A class of examples satisfying all the conditions imposed so far is afforded by Markov chains on the integers with transition probabilities of the form

$$p_{j,j+1} = \frac{1}{2}\left[1 + \frac{\alpha}{j} + o\left(\frac{1}{j}\right)\right] > 0,\ p_{j,j-1} = 1 - p_{j,j+1} \quad \text{if } j \neq 0; \tag{4}$$

$$p_{01} = 1 - p_{00} > 0.$$

For such chains (random walks) the *null* condition is known to hold if $\alpha > -1/2\ (= -\beta/2)$. In many (but, so far at least, not all) other cases, it can be shown that (3) follows automatically from the other hypotheses.

For a process $\{X_n\}$ satisfying the above assumptions, there is an analogue of the central-limit theorem:

---

[1] Partially supported by the National Science Foundation.

THEOREM 1.

$$\lim_{n\to\infty} \Pr(X_n \leq y(n)^{1/2}) = \int_0^y \frac{2\xi^{2\alpha/\beta}e^{-\xi^2/2\beta}}{(2\beta)^{\alpha/\beta+1/2}\Gamma\left(\frac{\alpha}{\beta}+\frac{1}{2}\right)}\,d\xi. \tag{5}$$

This seems to be a novel result even for random walks (despite the extensive recent development of their theory), and was reported in [**3**]. Under very slightly stronger hypotheses, however, much more is true. We shall call the process $\{X_n\}$ *uniformly null* provided the limit (3) holds uniformly in the initial state X_0. Again it can be shown that this often follows automatically; in particular, it holds for the random walks (4). For such processes we can prove

THEOREM 2. *For any $t>0$,*

$$\lim_{n\to\infty} \Pr(X_{[nt]} \leq y(n)^{1/2} \mid X_0 = x(n)^{1/2}) = p_t(x, y) \tag{6}$$

exists; the limit $p_t(x, y)$ is the transition-probability function for the diffusion process with backward equation

$$u_t = \frac{\alpha}{x}u_x + \frac{\beta}{2}u_{xx} \qquad (\alpha \text{ and } \beta \text{ are as in (2)}) \tag{7}$$

and with a reflecting barrier (if necessary) at the origin.

With the aid of these results it is easy to see that there is an analogue of the multi-dimensional C.L.T.; that is, the limit of

$$\Pr(X_{[nt_1]} \leq y_1(n)^{1/2}, \cdots, X_{[nt_k]} \leq y_k(n)^{1/2})$$

can be calculated. It is then natural to seek the appropriate version of the Erdös-Kac-Donsker *invariance principle* [**1**]. Define a continuous function $x_t^{(n)}$ by setting

$$x_t^{(n)} = \frac{X_i}{n^{1/2}} \quad \text{when} \quad t = \frac{i}{n}, \qquad i \leq n, \tag{8}$$

and by linear interpolation for other t. Let C be the space of all continuous functions x_t on $[0, 1]$ with $x_0=0$, and endow C with the uniform topology. Our main result is

THEOREM 3. *Under the conditions of Theorem 2,*

$$\lim_{n\to\infty} \Pr(f(x_t^{(n)}) \leq \alpha) = \Pr(f(x_t) \leq \alpha), \tag{9}$$

where x_t is the diffusion process encountered in Theorem 2, and where $f(\ )$ is a functional on C continuous almost everywhere with respect to the measure of the process $\{x_t\}$.[2]

From this a large number of interesting limit theorems follow (as from Donsker's theorem) by choosing specific functionals $f(\cdot)$. An important example for which the limit distribution can be obtained more or less explicitly is the case $f(x_t)=\max\{x_t|0\leqq t\leqq 1\}$.

Theorems 1 and 2 are reminiscent of a general limit theorem in Khintchine [2], and Theorem 3 of recent work of Prokhorov [4] and Skorohod [5]. None of these general results seem to be directly useful in proving the above theorems, however. Our proofs, together with additional results and applications, some extensions, and more complete references, will be published separately in the near future. It might be remarked that the methods are, for the most part, quite elementary. Calculations with moments and use of the moment-convergence theorem are prominent in the proofs of Theorems 1 and 2, while that of Theorem 3 is analogous in large measure to Donsker's procedure in [1].

References

1. M. Donsker, *An invariance principle for certain probability limit theorems*, Memoirs Amer. Math. Soc. no. 6, 1951.

2. A. Khintchine, *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, New York, Chelsea, 1948 (reprinted).

3. J. Lamperti, *Limit theorems for certain stochastic processes*, Abstract 569-32 Notices Amer. Math. Soc. vol. 7 (1960) pp. 268–269.

4. Yu. V. Prokhorov, *Convergence of random processes and limit theorems in probability theory*, Teor. Veroyatnost. i Primenen vol. 1 (1956) pp. 289–319.

5. A. V. Skorohod, *Limit theorems for Markov processes*, Teor. Veroyatnost. i Primenen vol. 3 (1958) pp. 217–264.

Stanford University

[2] It is understood that the diffusion process satisfies the initial condition $x_0=0$ and that the convergence in (9) is for all α for which the right-hand side is continuous.

THE GENERALISED POINCARÉ CONJECTURE

BY E. C. ZEEMAN

Communicated by Edwin Moise, February 3, 1961

THEOREM. *If a combinatorial n-manifold has the homotopy-type of an n-sphere then it is homeomorphic to an n-sphere, provided* $n \geqq 5$.

The above theorem was proved for $n \geqq 7$ by Stallings [2]. His proof can be adapted to cover the cases $n = 5, 6$ by means of the following lemma (the proof of which is given in [3]).

LEMMA. *Suppose* M^n *is a q-connected combinatorial n-manifold, where* $q \leqq n-3$. *Suppose* A^q *is a q-subcomplex, and B a collapsible subcomplex, both contained in the interior of* M^n. *Then there exists a collapsible subcomplex C in the interior of a suitable subdivision* σM^n *of* M^n, *such that* $C \supset \sigma(A^q + B)$ *and* $\dim(C - \sigma B) \leqq q+1$.

The lemma is useful in a variety of contexts. For the application that we need here, choose A^q to be the q-skeleton of M^n and B to be a point; then a regular neighbourhood of C is an n-ball containing A^q. Therefore if there are complementary skeletons of M^n with codimension at least 3, we can embed them in balls, and so, by expanding one of the balls, cover M^n by two balls. The theorem follows as in [2, Lemma 3]. Complementary skeletons of codimension at least 3 exist if and only if $n \geqq 5$.

In dimensions $n = 3, 4$ there is not quite enough elbow room for the proof to work, and so these two dimensions are the only outstanding cases for which the combinatorial form of Poincaré's conjecture remains open.

The combinatorial theorem above implies the analogous differential theorem of Smale [1], because differentiable manifolds can be triangulated, but not conversely.

BIBLIOGRAPHY

1. S. Smale, *The generalised Poincaré conjecture in higher dimensions*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 373–375.

2. J. R. Stallings, *Polyhedral homotopy-spheres*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 485–488.

3. E. C. Zeeman, *Isotopies of manifolds*, to appear.

GONVILLE AND CAIUS COLLEGE,
CAMBRIDGE, ENGLAND

DERIVATIONS OF COMMUTATIVE BANACH ALGEBRAS

BY PHILIP C. CURTIS, JR.[1]

Communicated by John W. Green, January 11, 1961

In [2] Singer and Wermer showed that a bounded derivation in a commutative Banach algebra $\mathfrak{A}$ necessarily maps $\mathfrak{A}$ into the radical $\mathfrak{R}$. They conjectured at this time that the assumption of boundedness could be dropped. It is a corollary of results proved below that if $\mathfrak{A}$ is in addition regular and semi-simple, this is indeed the case.

What is actually proved here is that under the above hypotheses, if D is a derivation of $\mathfrak{A}$ into $C(\Phi_{\mathfrak{A}})$,[2] $\Phi_{\mathfrak{A}}$ the structure space of $\mathfrak{A}$, then D is a bounded operator from $\mathfrak{A}$ to $C(\Phi_{\mathfrak{A}})$. The topologies are the norm topology in $\mathfrak{A}$ and the sup norm topology in $C(\Phi_{\mathfrak{A}})$. An application of the closed graph theorem shows that if D maps $\mathfrak{A}$ into itself, D must be a bounded operator in $\mathfrak{A}$, hence by the Singer, Wermer theorem, $D=0$.

If $\mathfrak{A}$ is regular but not semi-simple, then it follows from the above that D will map $\mathfrak{A}$ into $\mathfrak{R}$ provided that D maps $\mathfrak{R}$ into $\mathfrak{R}$. This the author can verify only if $\mathfrak{R}$ is nilpotent.

In what follows $\mathfrak{A}$ will always denote a regular, commutative, semi-simple Banach algebra with norm $\|\cdot\|$. Applying the Gelfand isomorphism we will identify $\mathfrak{A}$ and the corresponding subalgebra of $C(\Phi_{\mathfrak{A}})$. For convenience we also will assume $\mathfrak{A}$ possesses an identity. It is easily seen that this doesn't affect the generality of the results.

Let $\mathfrak{M}_\phi$ be a maximal ideal of $\mathfrak{A}$, and ϕ the corresponding point in $\Phi_{\mathfrak{A}}$. It is noted in [2] that there exists a derivation D of $\mathfrak{A}$ into some semi-simple extension $\mathfrak{B}$ of $\mathfrak{A}$ iff $\mathfrak{M}_\phi^2 \neq \mathfrak{M}_\phi$ for some maximal ideal $\mathfrak{M}_\phi$. In fact $\mathfrak{B}$ may be taken to be $B(\Phi_{\mathfrak{A}})$, the ring of bounded complex functions on $\Phi_{\mathfrak{A}}$. For if this condition is satisfied, following Singer and Wermer, we define by Zorn's Lemma a nontrivial linear functional f_ϕ on $\mathfrak{A}$ which annihilates $\mathfrak{M}_\phi^2$ and the identity. If we define D by

$$\begin{aligned} Dx(\phi') &= 0, \qquad \phi' \in \Phi_{\mathfrak{A}}, \qquad \phi' \neq \phi, \qquad\qquad x \in \mathfrak{A}, \\ Dx(\phi) &= f_\phi(x), \end{aligned}$$

it is easily seen that D is a derivation of $\mathfrak{A}$ into $B(\Phi_{\mathfrak{A}})$. D is in general unbounded, but if $\overline{\mathfrak{M}_\phi^2} \neq \mathfrak{M}_\phi$, f_ϕ, and consequently D, may be chosen (via the Hahn-Banach Theorem) to be bounded. Modifying the

[1] This research was supported by the United States Air Force, Office of Scientific Research, under contract AF49(638)-859.

[2] $C(\Phi_{\mathfrak{A}})$ denotes the algebra of continuous complex functions on the space $\Phi_{\mathfrak{A}}$.

terminology of Singer and Wermer somewhat we refer to both the functionals f_ϕ and the associated operators D as point derivations. The main result of this note is that any derivation D of $\mathfrak{A}$ into $B(\Phi_{\mathfrak{A}})$ is the sum of a bounded derivation and finitely many unbounded point derivations.

The key to the argument is the following result from [1, §3] stated in a form suitable to our needs.

THEOREM 1. *Let* $\|\cdot\|_1$ *be a norm on* $\mathfrak{A}$ *under which* $\mathfrak{A}$ *is a normed algebra. Let* $\mathcal{G}$ *be the class of open sets* G *for which there exist constants* M_G *satisfying*

$$\|x\|_1 \leqq M_G\|x\|, \qquad x \in \mathfrak{A}; \qquad c(x) \subset G.^3$$

Then there exists a finite subset F *of* $\Phi_{\mathfrak{A}}$, *called the singularity set of the norm* $\|\cdot\|_1$, *with the following two properties*:

(1) *If* G *is open and* $\overline{G}\cap F=\emptyset$, *then* $G\in\mathcal{G}$.
(2) *If* $G\in\mathcal{G}$, *then* $G\cap F=\emptyset$.

We now state and prove the result of the note.

THEOREM 2. *Let* D *be a derivation of* $\mathfrak{A}$ *into* $B(\Phi_{\mathfrak{A}})$. *Then there exists a finite subset* F *of* $\Phi_{\mathfrak{A}}$ *and a bounded derivation* D_1 *of* $\mathfrak{A}$ *into* $B(\Phi_{\mathfrak{A}})$ *such that if* $D_2=D-D_1$, *then* $D_2x(\phi)=0$, $x\in\mathfrak{A}$ *and* $\phi\in\Phi_{\mathfrak{A}}-F$. *For* $\phi\in F$, $f_\phi(x)\equiv D_2x(\phi)$ *is an unbounded point derivation. If for each* $x\in\mathfrak{A}$, $Dx\in C(\Phi_{\mathfrak{A}})$, *then* $F=\emptyset$ *and* D *is a bounded operator.*

PROOF. Re-norm $\mathfrak{A}$ by defining for $x\in\mathfrak{A}$ $\|x\|_1=\|x\|+\|Dx\|_\infty$ where $\|y\|_\infty=\sup_{\phi\in\Phi_{\mathfrak{A}}}|y(\phi)|$. Clearly $\mathfrak{A}$ is a normed algebra under $\|\cdot\|_1$. Therefore if F is the singularity set for $\|\cdot\|_1$, we assert $f_\phi(x)\equiv Dx(\phi)$ is a bounded linear functional on $\mathfrak{A}$ iff $\phi\notin F$. If $\phi\notin F$, then by the regularity of $\mathfrak{A}$ there exists $h_\phi\in\mathfrak{A}$ and a neighborhood V of F such that $h_\phi(\phi)=1$, $h_\phi(V)=0$. Let $\mathfrak{J}_V=\{x\in\mathfrak{A}: x(V)=0\}$. Choose an open set W, $\overline{W}\cap F=\emptyset$ such that if $x\in\mathfrak{J}_V$, then $c(x)\subset W$. Then by Theorem 1, D is bounded on $\mathfrak{J}_V$. Hence if $\{x_n\}$ is any sequence in $\mathfrak{A}$ tending to zero, then $x_nh_\phi\in\mathfrak{J}_V$ and $x_nh_\phi\to 0$. Consequently $D(x_nh_\phi)\to 0$. But $D(x_nh_\phi)(\phi)=Dx_n(\phi)+x_n(\phi)\cdot Dh_\phi(\phi)$. Therefore $f_\phi(x_n)\equiv Dx_n(\phi)\to 0$. For the converse let $H=\{\phi: f_\phi$ is bounded on $\mathfrak{A}\}$. Since $\|Dx\|_\infty<\infty$ for each $x\in\mathfrak{A}$, there exists by the principle of uniform boundedness, a constant M such that $\sup_{\phi\in H}|Dx(\phi)|\leqq M\|x\|$. If $\phi_0\in H\cap F$, pick an open set $G\subset H$, $\phi_0\in G$ and an element $y\in\mathfrak{A}$ for which $y(G)=1$ and $y(\Phi_{\mathfrak{A}}-H)=0$. Then if $x\in\mathfrak{A}$ and $c(x)\subset G$, we have $xy=x$. Therefore $Dx=yDx+xDy$ and

[3] $c(x)$ denotes the carrier of the function x.

$$\|Dx\|_\infty \leqq \sup_{\phi\in H} |y(\phi)\cdot Dx(\phi)| + \|x\|\cdot\|Dy\|_\infty$$
$$\leqq \{\|y\|_\infty\cdot M + \|Dy\|_\infty\}\|x\|.$$

This contradicts property (2) of Theorem 1.

If $F\neq\varnothing$ and D is unbounded, we define D_1 by

$$D_1x(\phi) = Dx(\phi), \qquad \phi\notin F,$$
$$= 0, \qquad \phi\in F.$$

Again applying the uniform boundedness principle it follows that D_1 is a bounded operator from $\mathfrak{A}$ to $B(\Phi_{\mathfrak{A}})$. The statement about D_2 is clear.

To complete the proof we observe first that if ϕ is isolated in $\Phi_{\mathfrak{A}}$, then $\phi\notin F$. In fact for such ϕ, $Dx(\phi)=0$. For let k_ϕ be the characteristic function of $\{\phi\}$. Then $k_\phi\in\mathfrak{A}$ and for $x\in\mathfrak{A}$ $D(k_\phi x)(\phi)=0$. Hence $Dx(\phi)=-x(\phi)\cdot Dk_\phi(\phi)=0$. Consequently $\overline{\Phi_{\mathfrak{A}}-F}=\Phi_{\mathfrak{A}}$. Therefore if for each $x\in\mathfrak{A}$, Dx is a continuous function on $\Phi_{\mathfrak{A}}$, it follows that $\|Dx\|_\infty=\sup_{\phi\in\Phi_{\mathfrak{A}}-F}|Dx(\phi)|\leqq M\|x\|$. This completes the proof.

Corollary. *Let $\mathfrak{B}$ be a subalgebra of $C(\Phi_{\mathfrak{A}})$ containing $\mathfrak{A}$. If $\mathfrak{B}$ is a Banach algebra under some norm and D is a derivation of $\mathfrak{A}$ into $\mathfrak{B}$, then D is a bounded operator. If D maps $\mathfrak{A}$ into itself, then $D\equiv 0$.*

Proof. The first result follows by the closed graph theorem. An application of the theorem of Singer and Wermer [2] then yields the second.

If now $\mathfrak{A}$ is not semi-simple and D maps $\mathfrak{A}$ into itself, then one may factor out the radical and apply the above corollary to prove that D maps $\mathfrak{A}$ into $\mathfrak{R}$ provided that D maps $\mathfrak{R}$ into $\mathfrak{R}$. If $\mathfrak{R}$ is nilpotent, this follows. For if $x^n=0$, then $0=D^nx^n=n!(Dx)^n+$terms each of which involves a positive power of x, hence belongs to the radical. Therefore $(Dx)^n\in R$, and consequently $Dx\in R$.

The validity of this result for non-nilpotent radicals is unknown to the author. Without some topological assumptions the result is of course false. Ordinary differentiation in the ring of formal power series is a derivation which does not map the radical into itself.

Bibliography

1. W. G. Bade and P. C. Curtis, Jr., *Homomorphisms of commutative Banach algebras*, Amer. J. Math. vol. 82 (1960) pp. 589–608.

2. I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. vol. 129 (1955) pp. 260–264.

University of California, Los Angeles

SLENDER GROUPS

BY R. J. NUNKE[1]

Communicated by R. S. Pierce, February 9, 1961

Let P be the direct product of countably many copies of the integers Z, i.e., the group of all sequences $x=(x_1, x_2, \cdots)$ of integers with term-wise addition; and, for each natural number n, let δ^n be the element in P whose nth coordinate is 1 and whose other coordinates are 0. Łoś calls a torsion-free abelian group A *slender* if every homomorphism of P into A sends all but a finite number of the δ^n into 0. The concept first appeared in [3]. E. Sąsiada [6] has shown that all reduced countable groups are slender. In this note I give a new description of the slender groups and apply it to show that certain classes of groups are slender. All groups in this paper are abelian.

A group is slender if and only if every homomorphic image of P in it is slender. It is therefore desirable to know the structure of the homomorphic images of P.

THEOREM 1. *A homomorphic image of P is the direct sum of a divisible group, a cotorsion group, and a group which is the direct product of at most countably many copies of Z.*

A group A is a *cotorsion* group if it is reduced and is a direct summand of every group E containing it such that E/A is torsion-free. These groups were introduced by Harrison [4]. A special case of Theorem 1 (namely the structure of P/S where S is the direct sum) was proved by S. Balcerzyk [2].

A torsion-free cotorsion group contains a copy of the p-adic integers for some prime p. For each prime p the p-adic integers are not slender: the homomorphism $x \rightarrow \sum_{i=1}^{\infty} x_i p^i$ sends δ^i into p^i. Theorem 1 and the remark preceding it then give

THEOREM 2. *A torsion-free group is slender if and only if it is reduced, contains no copy of the p-adic integers for any prime p, and contains no copy of P.*

A group is called $\aleph_1$-free if every at most countable subgroup is free.

COROLLARY 3. *An $\aleph_1$-free group is slender if and only if it contains no copy of P.*

[1] This work was supported by the National Science Foundation research contract N.S.F.—G11098.

A group A is a *B-group* if $\operatorname{Ext}(A, T)=0$ for every torsion group T and a *W-group* if $\operatorname{Ext}(A, Z)=0$. The names for these classes of groups are due to J. J. Rotman. All B-groups and W-groups are $\aleph_1$-free. Baer showed in [1] that P is not a B-group. It is also true that P is not a W-group. Since every subgroup of a B-group (W-group) is a B-group (W-group) we have

THEOREM 4. *Every B-group and every W-group is slender.*

This theorem was first proved (with an additional condition on the B-groups) by Rotman [5].

The above scheme can be applied to various other classes of groups, for example the torsion-free groups such that $\operatorname{Ext}(A, Z)$ is countable. The property is hereditary, every such group is $\aleph_1$-free, and P is not one of them. The structure of $\operatorname{Ext}(P, Z)$ is completely known. Let Q be the additive group of rational numbers and c the cardinal of the continuum.

THEOREM 5. $\operatorname{Ext}(P, Z)$ *is the direct sum of* 2^c *copies of* Q *and* 2^c *copies of* Q/Z.

REFERENCES

1. R. Baer, *Die Torsionsuntergruppe einer abelschen Gruppe*, Math. Ann. vol. 135 (1958) pp. 219–234.

2. S. Balcerzyk, *On factor groups of some subgroups of a complete direct sum of infinite cyclic groups*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys. vol. 7 (1959) pp. 141–142.

3. L. Fuchs, *Abelian groups*, Budapest, Publishing House of the Hungarian Academy of Sciences, 1958.

4. D. K. Harrison, *Infinite Abelian groups and homological methods*, Ann. of Math. vol. 69 (1959) pp. 366–391.

5. J. Rotman, *On a problem of Baer and a problem of Whitehead*, to appear.

6. E. Sąsiada, *Proof that every countable and reduced torsion-free Abelian group is slender*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys. vol. 7 (1959) pp. 143–144.

UNIVERSITY OF WASHINGTON

CONVERGENCE OF STOCHASTIC PROCESSES

BY V. S. VARADARAJAN[1]

Communicated by J. L. Doob, December 30, 1960

1. Many problems in probability theory, when properly formulated, appear as problems in the theory of convergence of stochastic processes. The need for such a theory was demonstrated by the early results of Doob [4], Donsker [5] and others. In their fundamental papers, LeCam [10] and Prohorov [11] developed several aspects of such a theory. Their work was based on, and was a development of, the earlier work of A. D. Alexandrov [1] and Kolmogorov [9]. However, several questions which naturally arise were either not discussed or discussed only under unnecessary restrictions. The following remarks contain an outline of a general theory of measures on topological spaces. Only the statements and the appropriate formulations of the main results are given. The detailed proofs will be published elsewhere.

2. Let X be a topological space and $C(X)$ the Banach space of bounded real-valued continuous functions on X. S is the smallest σ-field of subsets of X with respect to which all the elements of $C(X)$ are measurable. By measure we mean probability measures defined on S and these arise, in the classical manner following F. Riesz, from linear functionals ϕ defined on $C(X)$. Given a nonnegative linear functional ϕ on $C(X)$ with $\phi(1)=1$, we have the representation

$$\phi(f) = \int_X f d\mu \tag{1}$$

for all $f \in C(X)$ with a (unique) measure μ, provided ϕ is *σ-smooth*, i.e. for any sequence $\{f_n\}$ of elements of $C(X)$, $\downarrow 0$ pointwise over X, $\phi(f_n) \to 0$. The set of all measures is denoted by $M(X)$, or simply by M, when there is no doubt as to what X is.

M is a subset of the dual-space of $C(X)$ and as such inherits the weak topology of the dual of $C(X)$. Our main concern is with the structure of this topology over M and its subsets. The two main problems examined are the metrizability of M and the structure of compact subsets of M.

[1] This work was done during 1958–1959 while the author was in the Indian Statistical Institute, Calcutta, but due to diverse reasons the announcement was delayed up to now.

3. We begin with a classification of measures. A measure μ is called *τ-smooth*, if

$$\int_X f_j d\mu \to 0 \tag{2}$$

for every net[3] $\{f_j\} \downarrow 0$ pointwise over X. The set of all *τ-smooth* measures is denoted by M_τ. A measure μ is called *tight* if

$$\int_X \cdot\, d\mu$$

is continuous on the unit sphere of $C(X)$ with respect to the topology of uniform convergence on compacta. The set of tight measures on X is denoted by M_t. Clearly $M_t(X) \subset M_\tau(X) \subset M(X)$.

THEOREM 1. *In order that a measure μ be τ-smooth it is sufficient that there exists a closed Lindelöf[3] subset C of X such that $\mu_*(X-C)=0$, μ_* denoting the inner measure induced by μ. If X is paracompact, this condition is necessary and sufficient.*

In particular, if X is a metric space, μ is *τ-smooth* if and only if there exists a closed separable subset C of X such that $\mu(X-C)=0$.

It is interesting to examine the conditions under which we have the relation

$$M = M_\tau.$$

From Theorem 1 it follows at once that this is the case as soon as X is a separable metric space.

THEOREM 2. *If X is a metric space, $M(X) = M_\tau(X)$ if and only if $M(X_0) = M_\tau(X_0)$ for every closed discrete subspace X_0 of X.*

In other words $M(X) = M_\tau(X)$ if and only if the only measures defined on closed discrete subspaces of X are those with mass concentrated on a countable set. It is well known [2, p. 187] that this question is intimately related with some questions in the theory of sets. In particular, it follows, on assuming the continuum hypothesis, that $M(X) = M_\tau(X)$ for any metric space of cardinality less than or equal to that of the continuum.

THEOREM 3. *A measure μ is tight if and only if for each $\epsilon>0$ there exists a compact set $K_\epsilon \subset X$ such that*

$$\mu_*(X - K_\epsilon) < \epsilon.$$

[3] See [8] for this and other terminology concerning topological spaces.

If X is a complete metric space, $M_t = M_\tau$. In particular if X is a separable and complete metric space, $M = M_\tau = M_t$.

4. The questions of metrizability and compactness in M were first examined by P. Levy in the case when X is the real line (cf. for instance [6]) who proved that $M(X)$ in this case is a separable and complete[3] metric space. When X is an arbitrary topological space, an examination of the imbedding

$$\gamma: x \to \mu_x, \tag{3}$$

which sends $x \in X$ into the measure μ_x concentrated at x, reveals that for the metrizability of the space $M(X)$ one must have (i) X is metrizable and (ii) $M(X) = M_\tau(X)$. In view of this and Theorem 2 it is thus natural to attempt to prove that M_τ is metrizable whenever X is. We have

THEOREM 4. *If X is a metric space, $M_\tau(X)$ is metrizable. $M_\tau(X)$ is metrizable as a complete metric space when and only when X is a complete metric space.*

For the completeness part we note that the imbedding (3) of X into $M_\tau(X)$ sends X onto a closed subset of $M_\tau(X)$. On the other hand, if X is complete, we introduce βX, its Stone-Čech compactification ($X \subset \beta X$). By a general theorem due to Čech [3], X is a G_δ in βX. Any measure μ on X gives rise to a measure $\bar{\mu}$ on βX for which $\bar{\mu}(\beta X - X) = 0$. It can be shown that $\bar{\mu}$ is regular (in the sense of [7, p. 224]) if and only if $\mu \in M_\tau(X)$ and that the regular measures on βX are precisely the τ-smooth measures on βX. We thus obtain an imbedding of $M_\tau(X)$ into $M_\tau(\beta X)$. It can further be proved that this is a homeomorphism and that the image of $M_\tau(X)$ is a G_δ in $M_\tau(\beta X)$ (which is a compact Hausdorff space incidentally). Čech's theorem now assures us that $M_\tau(X)$ can be made complete under an equivalent metric.

We proceed next to a study of the compact subsets of M. In view of the nature of this communication we shall restrict ourselves to the case of greatest interest from the point of view of applications. A set $D \subset M(X)$ is called *tight* iff for each $\epsilon > 0$ there exists a compact set $K_\epsilon \subset X$ such that

$$\sup_{\mu \in D} \mu_*(X - K_\epsilon) < \epsilon, \tag{4}$$

It is clear that only subsets of $M_t(X)$ can be tight sets; further,

[3] Completeness, here as elsewhere, is always completeness under some equivalent metric.

it can be proved that a set $D \subset M_t$ is tight if and only if the corresponding set of linear functionals are equicontinuous at 0 in the topology (over the unit sphere of $C(X)$) of uniform convergence on compacta. The following theorem is easy to prove.

THEOREM 5. *If D is tight, then $\bar{D} \subset M_t$ and is compact.*

In [9] Kolmogorov raised the interesting question as to whether the converse of Theorem 5 is true. That this is so, when X is a separable and complete metric space, was proved by Prohorov [11]. The following theorem settles the question when X is an arbitrary metric space.

THEOREM 6. *If X is a metric space and D is a compact subset of M_t, then D is a tight set.*

In view of well-known Ascoli theorems [8, p. 233] the proof proceeds by showing that the map

$$(f, \mu) \to \int f d\mu$$

of $C_1(X)XD$ into the reals ($C_1(X)$ being the unit sphere of X under the topology of uniform convergence on compacta) is continuous. We note now that D is a metric space (Theorem 4) and hence convergence on D is of a sequential nature. The desired continuity is now obtained by using a theorem of LeCam [10] (which is essentially Theorem 6 for the case when D consists of a convergent sequence plus its limit point).

It might be remarked that when X is not a metric space there are examples of compact subsets D of M_t which are not tight.

5. Applications to stochastic processes arise when we regard a stochastic process as a measure on a topological space X of functions. If

$$\xi, \xi_1, \xi_2, \cdots \tag{5}$$

is a sequence of stochastic processes, the convergence of ξ_n, as $n \to \infty$, to ξ then implies the convergence of the distribution of $g[\xi_n]$ to that of $g[\xi]$ for all continuous functions g on X. Typical problems are those in which X is a separable Banach space and ξ is a random variable with values in X which is normally distributed i.e. for any bounded linear functional x^* on X, $x^*(\xi)$ is normally distributed. We then consider the central limit problem. Let

$$\eta_1, \eta_2, \cdots \tag{6}$$

be independent identically distributed X-valued random variables with $E(\eta_i)\equiv 0$. If

$$\xi_n = \frac{1}{n}(\eta_1 + \cdots + \eta_n),$$

under what conditions do the distributions of ξ_n converge in X? When X is a Hilbert space, Prohorov proved [11] that $E\|\eta_i\|^2 < \infty$ was a necessary and sufficient condition. The general problem when X is an arbitrary separable Banach space remains unsolved, but the following theorem is one of several special results:

THEOREM 7. *If X is the space l_1 (of all sequences*

$$\alpha = (a_1, a_2, \cdots)$$

with $\sum_n |a_n| < \infty$ for which $\|\alpha\| = \sum_n |a_n|$), then

$$\sum_n \left[\operatorname{Var} \eta_1^{(n)}\right]^{1/2} < \infty \tag{7}$$

is a necessary and sufficient condition that ξ_n should converge in distribution.

Here $\eta_1^{(n)}$ denotes the nth component of η_1 and Var denotes variance.

REFERENCES

1. A. D. Alexandrov, *Additive set functions in abstract spaces*, Mat. Sb. vol. 8 (1940) pp. 307–348; vol. 9 (1941) pp. 563–628; vol. 13 (1943) pp. 169–238.

2. G. Birkhoff, *Lattice theory*, Rev. ed., Amer. Math. Soc. Colloquium Publications, vol. 25, 1948.

3. E. Čech, *On bicompact spaces*, Ann. of Math. vol. 38 (1937) pp. 823–844.

4. J. L. Doob, *Heuristic approach to the Kolmogorov-Smirnov theorems*, Ann. Math. Statist. vol. 20 (1949) pp. 393–403.

5. M. Donsker, *Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems*, Ann. Math. Statist. vol. 23 (1952) pp. 277–281.

6. B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions for sums of independent random variables*, Reading, Addison-Wesley, 1954.

7. P. R. Halmos, *Measure theory*, Princeton, Van Nostrand, 1950.

8. J. L. Kelley, *General topology*, Princeton, Van Nostrand, 1955.

9. A. N. Kolmogorov and Yu. V. Prohorov, *Zufällige Funktionen und Grenzverteilungssätze, Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik*, Berlin, Deutscher Verlag der Wissenschaften, 1954, pp. 113–126.

10. L. LeCam, *Convergence in distribution of stochastic processes*, Univ. California Publ. Statist. vol. 2 (1957) pp. 207–236.

11. Yu. V. Prohorov, *Convergence of random processes and limit theorems in probability theory*, Teor. Veroyatnost. i Primenen. vol. 1 (1956) pp. 177–238.

THE UNIVERSITY OF WASHINGTON

A CONTINUOUS FUNCTION WITH TWO CRITICAL POINTS

BY NICOLAAS H. KUIPER[1]

Communicated by R. P. Boas, February 6, 1961

A real C^s-function $f: X \to \mathbf{R}$ on an n-dimensional C^s-manifold with $s \geqq 0$, is called *C^s-nondegenerate C^s-ordinary* at a point $p \in X$, in case a system of n C^s-coordinates (C^s-functions) $\phi_1, \cdots, \phi_n$ exists, which defines a C^s-diffeomorphism x of some neighborhood $V(p)$ of p into $\mathbf{R}^n$, and such that for some constant $\lambda_p > 0$

$$(1)\phi_i(p) = 0, \ i = 1, \cdots, n; \ \phi_n(q) = \lambda_p\{f(q) - f(p)\}$$

for $q \in V(p) \subset X$.

If C^s-coordinates and $\lambda_p > 0$ exist such that

$$\phi_i(p) = 0, \qquad i = 1, \cdots, n;$$

$$(2) \qquad -\sum_1^r \phi_i^2(q) + \sum_{r+1}^n \phi_i^2(q) = \lambda_p\{f(q) - f(p)\}$$

then the function is called *C^s-critical of index r and C^s-nondegenerate at p.*

A function which is C^s-nondegenerate at every point $p \in X$ is called a *C^s-nondegenerate function.*

We will restrict our considerations to the topological case $s = 0$ of continuous functions on topological manifolds and we will omit C^0 from the notation in the sequel. By *function we will mean continuous function*, etc.

A compact manifold without boundary is called a *closed* manifold. A nondegenerate function on a closed manifold has at least one critical point p_1 of index n and one critical point p_0 of index 0, corresponding respectively with the maximum and the minimum of the function. We prove the

THEOREM. *If X is a closed n-dimensional manifold and $f: X \to \mathbf{R}$ a continuous nondegenerate function with exactly two critical points, then X is homeomorphic to the n-sphere S^n.*[2]

---

[1] The author has a research grant from the National Science Foundation, NSF-G-13989.

[2] Reeb [2] proved the corresponding theorem for the differentiable case. Morse [1] proved that X is a homotopy-sphere, and he also has a proof of the theorem we present (unpublished as yet).

PROOF. A. *The local droppings* T_p. We place ourselves in the assumptions of the theorem and we call the function f "height." We consider a coordinate system for every point $p \in X$, obeying (1) or (2), but for which moreover the image $\kappa_p(V(p)) \subset \mathbf{R}^n$ is the open n-ball

$$r < 5, \tag{3}$$

where the "*polar coordinates*" r (radius) and ω (unit vector) are defined by

$$r = \left(\sum_j^n \phi_j^2\right)^{1/2}, \qquad \omega = (\phi_1/r, \phi_2/r, \cdots, \phi_n/r). \tag{4}$$

For any such coordinate system κ: $V(p) \to \mathbf{R}^n$ we also define the open set

$$U_t(p) = \{q \mid q \in V(p) \subset X, r(q) < t\}. \tag{5}$$

Next we define a homeomorphism T_p for every $p \in X$. If p is an *ordinary point* then we proceed as follows:

Let $h(t)$ be a real C^∞-function with the properties

$$\begin{aligned} h(t) &\begin{cases} = 0, & |t| \geqq 4, \\ > 0, & |t| < 4, \\ = h(0), & |t| \leqq 1, \end{cases} \\ |h'(t)| &< 1/2, \quad \text{any } t. \end{aligned} \tag{6}$$

The homeomorphism T_p is given by:

$$\begin{aligned} &\left.\begin{aligned} \phi_i(T_p(q)) &= \phi_i(q), \quad i = 1, \cdots, n-1 \\ \phi_n(T_p(q)) &= \phi_n(q) - h(r(q)) \end{aligned}\right\} && q \in V(p), \\ &T_p(q) = q, && q \notin V(p). \end{aligned} \tag{7}$$

As the Jacobian of the corresponding C^∞-transformation of the coordinates for $q \in V(p)$ does not vanish, and $T(q) = q$ for $q \notin U_4(p)$, it follows that T_p is a global homeomorphism of X. Observe that the continuous function

$$q \to f(T_p(q)) - f(q): X \to \mathbf{R}$$

takes the value zero for $q \notin U_4(p)$ and is negative for $q \in U_4(p)$. It takes a negative maximal value on the set $\overline{U_3(p)}$, the closure in X of $U_3(p)$. Under T_p no point is mapped into a higher level of f, and every point of $U_4(p)$ is mapped into a lower level.

If p is a *critical point of index* n we use a real C^∞-function $k(t)$ with the properties

$$(8)\qquad k(t)\begin{cases} = t, & \text{for } t \geqq 4,\\ = 2t, & 0 \leqq t \leqq 1,\\ > t, & 0 \leqq t < 4,\end{cases}$$
$$k'(t) > 0, \qquad t \geqq 0.$$

The homeomorphism T_p is now defined in terms of polar coordinates (4) by:

$$(9)\qquad \left.\begin{aligned} \omega(T_p(q)) &= \omega(q)\\ r(T_p(q)) &= k(r(q))\end{aligned}\right\} \quad \text{for } q \in V(p),$$
$$T_p(q) = q \qquad \text{for } q \notin V(p).$$

The restriction of T_p to $U_1(p)$ is represented by a geometrical multiplication with factor 2 in coördinate space.

The point p and every point $q \notin U_4(p)$ is invariant under T_p. Every other point in X is mapped into a lower level.

In the case of *critical point of index zero* we use the function k^{-1}, the inverse of k, and proceed analogously.

B. *The global dropping* T. Under the given assumptions there is a critical point p_1 of index n (maximum), a critical point p_0 of index 0 (minimum), and no other critical point. Choose a finite number of coordinate systems κ_{p_i} and homeomorphisms T_{p_i}, $i=0, \cdots, L$, of the kinds mentioned above, such that:

$$(10)\qquad \bigcup_{i=0}^{L} U_2(p_i) = X$$

but

$$(10)\qquad \bigcup_{i=2}^{L} U_4(p_i) \cap [U_2(p_0) \cup U_2(p_1)] = \emptyset \text{ (void)}.$$

(Compare the use of a partition of unity.)

Let

$$(11)\qquad T = T_{p_L}T_{p_{L-1}} \cdots T_{p_2}T_{p_1}T_{p_0}.$$

Then $T: X \to X$ *is a global homeomorphism with exactly two invariant points, namely* p_0 *and* p_1, *which maps every other point into a lower level:*

$$(12)\quad T(p_0) = p_0;\ \ T(p_1) = p_1;\ \ f(T(q)) \leqq f(q) \quad \text{for } q \in X - p_0 - p_1.$$

As the set $W_\epsilon = X - U_2(p_1) - U_\epsilon(p_0)$ for $0<\epsilon<1$, is compact, the non-negative function

$$f(q) - f(T(q))$$

has a minimal value for $q \in W_\epsilon$ and this minimal value is positive. Call it $\delta_\epsilon > 0$ and let N_ϵ be an integer such that

$$N_\epsilon \delta_\epsilon > f(p_1) - f(p_0). \tag{13}$$

If we apply powers with consecutive exponents of the homeomorphism T, to any point $q \in W_\epsilon$, then for some exponent $N \leqq N_\epsilon$ we will find

$$T^N(q) \in U_\epsilon(p_0)$$

because with each new application of T to the result obtained in the last step, we obtain a new point which is at a level at least δ_ϵ lower, and after N_ϵ steps the point would have dropped totally more than the total range of the function f over X. On the other hand, once the resulting point is in $U_\epsilon(p_0)$ any further application of T will give a new point also in $U_\epsilon(p_0)$, because T acts in $U_\epsilon(p_0)$ as a geometrical multiplication with factor 1/2. Consequently

$$T^{N_\epsilon}(X - U_2(p_1)) \subset U_\epsilon(p_0) \tag{14}$$

and taking complements

$$T^{N_\epsilon}(U_2(p_1)) \supset X - U_\epsilon(p_0). \tag{14)c}$$

Thus X is covered by two discs:

$$T^{N_\epsilon}(U_2(p_1)) \cup U_\epsilon(p_0) = X \tag{15}$$

and our theorem can be considered as a consequence of a theorem of Morton Brown. However, we like to present a complete explicit proof:

C. *The homeomorphism $X \to S^n$.*

As (14) holds for any $0<\epsilon<1$, it follows that for any $q \neq p_0$ there exists a smallest number N_q such that

$$T^{N'} U_2(p_1) \ni q \qquad \text{for } N' \geqq N_q$$

or

$$T^{-N'}(q) \in U_2(p_1). \tag{16}$$

Let $\kappa_1\colon U_2(p_1) \to \mathbf{R}^n$ be the restriction of the coordinate system at the critical point p_1 to the open set $U_2(p_1)$. Observe that for any $q \in U_2(p_1)$:

$$2^k \cdot \kappa_1[T^{-k}(q)] = \kappa_1(q), \qquad k \geqq 0. \tag{17}$$

If $N' \geqq N = N_q$ then in view of (17) we have in the vector space $\mathbf{R}^n$:

$$2^{N'}\kappa_1[T^{-N'}(q) = 2^N \cdot 2^{N'-N}k_1(T^{-N'+N}T^{-N}q) = 2^N\kappa_1(T^{-N}q).$$

Hence there exists a mapping κ: $(X - p_0) \to \mathbf{R}^n$ well defined by:

$$\kappa(q) = 2^{N'}\kappa_1(T^{-N'}q), \qquad N' \geqq N_q. \tag{18}$$

κ is clearly locally a homeomorphism. κ is onto the set $\bigcup_{j=0}^{\infty} 2^j\kappa_1(U_2(p_1)) = \mathbf{R}^n$. If q_1 and $q_2 \neq q_1$ are both different from p_0 then, for $N' \geqq N_{q_1} + N_{q_2}$,

$$T^{-N'}(q_1) \neq T^{-N'}(q_2)$$

and consequently $\kappa(q_1) \neq \kappa(q_2)$. So κ: $(X - p_0) \to \mathbf{R}^n$ is a homeomorphism and X is homeomorphic to the one point compactification of $\mathbf{R}^n$, that is S^n.

Bibliography

1. Marston Morse, *Topologically non-degenerate functions on a compact n-manifold M*, J. Analyse Math. vol. 7 (1959) pp. 189–208.

2. G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Sci. Ind. No. 1183 (1952) pp. 91–154.

Landbouwhogeschool, Wageningen, Netherlands and
Northwestern University

TOPOLOGICAL EQUIVALENCE OF A BANACH SPACE WITH ITS UNIT CELL

BY VICTOR KLEE[1]

Communicated by Mahlon M. Day, January 3, 1961

Several years ago [8] we proved that Hilbert space is homeomorphic with both its unit sphere $\{x: \|x\|=1\}$ and its unit cell $\{x: \|x\| \leqq 1\}$. Later [9] we showed that in every infinite-dimensional normed linear space, the unit sphere is homeomorphic with a (closed) hyperplane and the unit cell with a closed halfspace. It seems probable that every infinite-dimensional normed linear space is homeomorphic with both its unit sphere and its unit cell, but the question is unsettled even for Banach spaces. Corson [4] has recently proved that every $\aleph_0$-dimensional normed linear space is homeomorphic with its unit cell. In the present note, we establish the same result for a class of infinite-dimensional Banach spaces which is *believed* to include *all* such spaces. It is *proved* to include every infinite-dimensional Banach space which is reflexive, or admits an unconditional basis, or is a separable conjugate space, or is a space CM of all bounded continuous real-valued functions on a metric space M.

We employ the following tools:

(1) If E and F are Banach spaces and u is a continuous linear transformation of E onto F, then there exist a constant $m \in]0, \infty[$ and continuous mapping v of F into E such that $uvx=x$, $vrx=rvx$, and $\|vx\| \leqq m\|x\|$ for all $x \in F$ and $r \in R$ (the real number space). If G is the kernel of u and $hy=(uy, vuy-y) \in F \times G$ for each $y \in E$, then h is a homeomorphism of E onto $F \times G$. Let $\|(p, q)\| = \max(\|p\|, \|q\|)$ for all $(p, q) \in F \times G$, and let $\xi y = (\|y\|/\|hy\|)hy$ for all $y \in E$. Then ξ is a homeomorphism of E onto $F \times G$ which carries the unit cell of E onto that of $F \times G$.

(2) If S is a closed linear subspace of a Banach space E, then E is homeomorphic with the product space $(E/S) \times S$ and the unit cell of E is homeomorphic with the unit cell of this product space (with respect to any norm compatible with the product topology).

(3) In each infinite-dimensional normed linear space, the unit cell is homeomorphic with a closed halfspace.

(4) If Q is an open halfspace in an infinite-dimensional normed linear space and p is a point in the boundary of Q, then $Q \cup \{p\}$ is homeomorphic with Q.

---

[1] Research Fellow of the Alfred P. Sloan Foundation.

(5) For each $f \in L^2]0, \infty[$ and $t \in [0, 1[$, let the function $f_t \in L^2]0, \infty[$ be defined as follows: $f_t x = tf(tx)$ for $x \in]0, 1[$; $f_t x = f(x+t-1)$ for $x \in [1, \infty[$. Then with $\eta(f, t) = (f_t, t)$, the transformation η is a homeomorphism of $L^2]0, \infty[\times[0, 1[$ onto $(L^2]0, \infty[\times]0, 1[) \cup (L^2[1, \infty[\times\{0\})$.

The existence of v and m as described in (1) follows from a theorem of Bartle and Graves [1, p. 404] (see also Michael [13]). It is easily verified that h is a homeomorphism [10], and homogeneity of h follows from that of u and v. Thus the transformation ξ is also homogeneous. To complete the proof of (1) it suffices to observe that $(1+m)^{-1}\|y\| \leq \|hy\| \leq (m\|u\|+1)\|y\|$ for all $y \in E$. Proposition (2) results from applying (1) to the canonical mapping u of E onto E/S.

The result (3) appears in [9]. For (5), see page 29 of [8]. A theorem much stronger than (4) is proved on pages 12–28 of [8]. When the space is nonreflexive or is an (l^p) space, (4) is explicitly a corollary of (3.3) on page 27 of [8]. In the general case, it follows from the reasoning (though not explicitly from any statement) in [8]. Also, a proof of (4) is outlined in [11].

A normed linear space J will be called *compressible* provided the space $J\times[0, 1[$ is homeomorphic with the space $(J\times]0, 1[) \cup (W\times\{0\})$ for some closed linear subspace W of infinite deficiency in J. (We see by (5) that Hilbert space is compressible.) A space is *h-compressible* provided it is homeomorphic with some compressible normed linear space.

THEOREM. *If a Banach space B admits a continuous linear transformation onto a Banach space E which contains an h-compressible closed linear proper subspace S, then B is homeomorphic with the unit cell of B.*

PROOF. Let G denote the kernel of the continuous linear transformation of B onto E. By (1), B is homeomorphic with the product space $P = E\times G$ and the unit cell of B is homeomorphic with the unit cell U of P. To establish the theorem, it suffices to show that P is homeomorphic with U. Since S is a closed linear proper subspace of E, the subspace $T = S\times\{0\}$ must be in a closed hyperplane V in P. The unit cell U of P is homeomorphic with $V\times[0, 1[$ by (3), and V is homeomorphic with $(V/T)\times T$ by (2), so U is homeomorphic with $(V/T)\times(T\times[0, 1[)$. Clearly P itself is homeomorphic with $V\times]0, 1[$ and hence with $(V/T)\times(T\times]0, 1[)$, so to complete the proof it suffices to show that $T\times[0, 1[$ is homeomorphic with $T\times]0, 1[$. Since T is h-compressible, there exist a Banach space J homeomorphic with T and a subspace W of infinite deficiency in J such that

$J\times[0,1[$ is homeomorphic with $(J\times]0,1[)\cup(W\times\{0\})$. Let u denote the canonical mapping of J onto J/W and then let v and h be as in (1) above. Then h is a homeomorphism of J onto $(J/W)\times W$, and since $hw=(\theta, v\theta-w)$ for all $w\in W$ (where θ is the neutral element of J/W), it follows that $hW=\{\theta\}\times W$. Consequently the space $(J\times]0,1[)\cup(W\times\{0\})$ is homeomorphic with

$$(J/W)\times W\times]0,1[\ \cup\ \{\theta\}\times W\times\{0\},$$

which in turn is homeomorphic with

$$W\times((J/W)\times]0,1[\ \cup\ \{0\}\times\{0\}).$$

Since J/W is infinite-dimensional, it follows by (4) that the set above is homeomorphic with

$$W\times((J/W)\times]0,1[),$$

and hence with $J\times]0,1[$. Reviewing the information now assembled, we see that $T\times[0,1[$ is homeomorphic with $T\times]0,1[$, and hence that U is homeomorphic with P. This completes the proof of the theorem.

Corollary. *If an infinite-dimensional Banach space B satisfies at least one of the following conditions, then B is homeomorphic with its unit cell*:

(a) *B is reflexive*;

(b) *B is a linear subspace of a Banach space which admits an unconditional basis*;

(c) *B is a norm-separable w^*-closed linear subspace of a conjugate space*;

(d) *B is the space CN of all bounded continuous real-valued functions on a normal space N which contains a closed infinite metrizable subset.*

Proof. In view of the theorem and the fact (by (5)) that Hilbert space is compressible, it suffices in each case to produce a continuous linear transformation of B onto a Banach space E which contains a closed linear proper subspace S which is homeomorphic with Hilbert space. When B is reflexive, let $E=B$ and let S be an infinite-dimensional separable closed linear proper subspace of E. Then S is reflexive and hence (by a theorem of Kadeč [7]) homeomorphic with Hilbert space.

If B is a subspace of a space which admits an unconditional basis, a theorem of James [5] and Bessaga and Pełczyński [2] asserts that either B is reflexive or some linear subspace of B is linearly homeomorphic with the space (l) or the space (c_0). But the latter two spaces are known to be homeomorphic with Hilbert space (by results of Mazur [12] and Kadeč [6]) and the desired conclusion follows.

Now suppose B is a separable conjugate space or, more generally, that B is a norm-separable w^*-closed linear subspace of a conjugate Banach space L^*. Let $f \in B \sim \{0\}$, $x \in L$ with $fx = 1$, and $S = \{g \in E: gx = 0\}$. Then S is a w^*-closed linear proper subspace of B, and must be homeomorphic with Hilbert space by a theorem in [10]. Consequently, B is homeomorphic with its unit cell.

Finally, let B and N be as in (d). Then there is a countably infinite closed subset Z of N which consists of either a discrete set or a convergent sequence together with its limit point. For each $\phi \in CN$ let $u\phi = \phi | Z \in CZ$. Then u is a continuous linear transformation of CN onto CZ, and CZ is equivalent to either the space (m) or the space (c_0). In either case, CZ has the h-compressible space (c_0) as a closed linear proper subspace, and the desired conclusion follows upon applying the theorem.

Note that the topological equivalence of every infinite-dimensional Banach space with its unit cell would be implied by the generally expected affirmative answer to the following question: Are all infinite-dimensional separable Banach spaces homeomorphic? Recent results on this problem have been obtained by Bessaga and Pełczyński [3].

At least for reflexive spaces, the corollary above can be significantly improved. The method is that of [8, pp. 30–31] in conjunction with the above techniques and the result is as follows:

THEOREM. *Suppose E is an infinite-dimensional reflexive Banach space and C is a closed convex subset of E which has nonempty interior. Then C is homeomorphic with E and the boundary of C is homeomorphic with E or with $E \times S^n$ for some finite n and n-sphere S^n.*

The following problems seem worthy of mention: Are all infinite-dimensional separable Banach spaces h-compressible? (An affirmative answer implies that every infinite-dimensional Banach space is homeomorphic with its unit cell.) Are all infinite-dimensional Banach spaces compressible? Are $\aleph_0$-dimensional normed linear spaces compressible? Note that for Hilbert space, the compressibility was achieved by means of a continuous family of affine homeomorphisms. How generally is this possible?

REFERENCES

1. R. G. Bartle and L. M. Graves, *Mappings between function spaces*, Trans. Amer. Math. Soc. vol. 72 (1952) pp. 400–413.

2. C. Bessaga and A. Pelczyński, *A generalization of results of R. C. James*, Studia Math. vol. 17 (1958) pp. 165–174.

3. ———, *Some remarks on homeomorphism of Banach spaces*, to appear.

4. H. H. Corson and Victor Klee, *Topological classification of convex bodies*, to appear.

5. R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. vol. 52 (1950) pp. 518–527.

6. M. I. Kadeč, *On homeomorphism of certain Banach spaces*, Dokl. Akad. Nauk SSSR vol. 92 (1953) pp. 465–468 (Russian).

7. ———, *On weak and norm convergence*, Dokl. Akad. Nauk SSSR vol. 122 (1958) pp. 13–16 (Russian).

8. Victor Klee, *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 10–43.

9. ———, *A note on topological properties of normed-linear spaces*, Proc. Amer. Math. Soc. vol. 7 (1956) pp. 673–674.

10. ———, *Mappings into normed linear spaces*, Fund. Math. vol. 49 (1960) pp. 25–34.

11. ———, *Topological structure of infinite-dimensional linear spaces*, to appear.

12. S. Mazur, *Une remarque sur l'homéomorphie des champs fonctionnels*, Studia Math. vol. 1 (1929) pp. 83–85.

13. Ernest Michael, *Continuous selections*. I, Ann. of Math. vol. 63 (1956) pp. 361–382.

University of Washington

ORTHOGONAL GROUPS OVER LOCAL RINGS

BY WILHELM KLINGENBERG

Communicated by J. Dieudonné, October 28, 1960

In an earlier paper [5] we have determined the structure of the linear groups over a local ring. In this note we continue the study of the classical groups over a local ring with the investigation of the orthogonal groups.

Our main result (cf. Theorem 6 below) is a complete description of the invariant subgroups of an orthogonal group of noncompact type (i.e., of index $\geqq 1$) over a local ring L of characteristic $\neq 2$. Certain low dimensional cases being excluded, the result reads as follows: The set of invariant subgroups splits into disjoint classes $\mathfrak{C}(J)$ which are in one-to-one correspondence with the ideals J of L. Each class has a greatest and a smallest element, with respect to the inclusion, which are represented by certain congruence subgroups modulo J, and every group between the greatest and the smallest element of $\mathfrak{C}(J)$ belongs to $\mathfrak{C}(J)$.

A similar result does hold for the set of invariant subgroups of the commutator group of the orthogonal group; in this case, the structure of the classes $\mathfrak{C}(J)$ is very simple since each class contains at most two elements, and then the smaller element has index 2 in the greater one.

Hence, it turns out that the structure of the orthogonal groups under consideration is of the same type as the structure of the linear groups over a local ring, cf. [5]: Here too the invariant subgroups split into classes which correspond to the ideals of the local ring, and each class has a greatest and a smallest element, represented by certain congruence subgroups, and each group in between belongs to the class. One may expect, therefore, that this is the typical arrangement of the invariant subgroups of a classical group over a local ring.

If the local ring L possesses no ideals apart from L and 0, i.e., if L is a field, then we get the results of Dieudonné [3; 4] on the structure of the orthogonal groups over a field.

1. **Basic definitions.** A *local ring* is a commutative ring L with unit and a greatest ideal $I \neq L$. $L^* = L - I$ forms a group under the multiplication. The homomorphic image of a local ring, if it is not the zero ring, is again a local ring. L/I is a field. *We assume*: $\mathrm{char}(L/I) \neq 2$.

An (*n-dimensional*) *vector space over* L is an L-module isomorphic to L^n. Let Φ be a symmetric bilinear form on a vector space. Φ determines an homomorphism g_Φ of the vector space into its dual, cf. Bourbaki [2]. Φ is called *nondegenerate*, if g_Φ is an isomorphism.

A *metric vector space* (*over* L), denoted by V or $V(L)$, is a vector space over L on which there is given a nondegenerate symmetric bilinear form Φ.

A *subspace* U *of* V is a submodule of V (considered as L-module), with the following properties: (i) U is a direct summand, (ii) kernel$(g_{\Phi|U}: U \to U^*)$ is a direct summand.

NOTE. If L is a field, each submodule of V is a subspace. To get uniform definitions and results *we assume*: dim $V \geqq 3$. For a subspace U the *orthogonal subspace* U° is the submodule annulled by $g_\Phi U$. U° is a subspace. We have: dim $U+$dim $U^\circ=$dim V. $U^{\circ\circ}=U$.

A subspace U is called *isotropic*, if kernel$(g_{\Phi|U})\neq 0$, and *totally isotropic*, if kernel $(g_{\Phi|U})=U$. Examples of nonisotropic subspaces (i.e., subspaces U with kernel$(g_{\Phi|U})=0$) are V and $0=0$-space.

A vector $X\in V$ is called *nonisotropic*, if the submodule $\langle X\rangle$, generated by X, is a nonisotropic subspace $\neq 0$. X is called *isotropic*, if $\langle X\rangle$ is isotropic. A nonisotropic vector X is characterized by the properties: $X\neq 0$ mod I and $\Phi(X, X)\in L^*$. An isotropic vector is characterized by: $X\neq 0$ mod I and $\Phi(X, X)=0$.

An isomorphism of a space V into a space V' is called *isometry*.

The group of isometries of V onto V is called *orthogonal group of* V, $\mathbf{O}(V)$. The subgroup of isometries with determinant one is called *special orthogonal group of* V, $\mathbf{SO}(V)$. We have: center $\mathbf{O}(V)=\{1, -1\}$. Let J be an ideal of L with $J\subset I$. The natural homomorphism $g_J: L\to L/J$ determines a *homomorphism* (also denoted by g_J)

$$(1) \qquad g_J: V(L)\to V(L/J),$$

where $V(L/J)$ is a space over the local ring L/J with a nondegenerate symmetric bilinear form $h_J\Phi$ characterized by: $(h_J\Phi)(g_JX,\ g_JY)=g_J\Phi(X, Y)$ for $(X, Y)\in V\times V$. We will permit in (1) also the ideal $J=L$ by putting $V(L/L)=0=0$-space.

The homomorphism g_J, (1), determines a *homomorphism*

$$(2) \qquad h_J: \mathbf{O}(V(L))\to \mathbf{O}(V(L/J))$$

with the characteristic property: $h_J\sigma g_J=g_J\sigma$ for $\sigma\in\mathbf{O}(V)$. Here $\mathbf{O}(V(L/L))$ denotes the unit group E.

The *congruence subgroup* mod J of $\mathbf{O}(V)$, $\mathbf{O}(V, J)$, is the invariant subgroup consisting of the elements $\sigma\in\mathbf{O}(V)$ with $h_J\sigma\in$center $\mathbf{O}(g_J V)$. $\mathbf{SO}(V)\cap\mathbf{O}(V, J)$ is called *special congruence subgroup* mod J *of* $\mathbf{SO}(V)$, notation: $\mathbf{SO}(V, J)$.

NOTE. $\mathbf{O}(V,\ L)=\mathbf{O}(V)$. $\mathbf{SO}(V,\ L)=\mathbf{SO}(V)$. $\mathbf{O}(V,\ O)=$center $\mathbf{O}(V)$. Here O denotes the zero ideal.

For each ideal J of L the *congruence commutator subgroup* mod J of $SO(V)$, $\Omega(V, J)$, is defined as the mixed commutator group comm$(SO(V), SO(V, J))$. $\Omega(V, L)$, i.e., the *commutator group of* $SO(V)$, will also be denoted by $\Omega(V)$. Note: $\Omega(V, O)=E$.

2. The theorems of Witt and Cartan-Dieudonné. A first characterization of the congruence commutator groups. We have the following theorem which reduces to the theorem of Witt if L is a field:

THEOREM 1. *Let V and V' be isometric spaces. If $\sigma: U\to V'$ is an isometry of a subspace U of V into V', then there exists an isometry of V onto V' which is an extension of σ.*

As a consequence we have that all maximal totally isotropic subspaces of V have the same dimension; this dimension is called the *index of* V, notation: ind V. We have: 2 ind $V\leqq\dim V$. If U is a non-isotropic subspace, then we have $V=U+U^*$ (direct sum). The *symmetry with respect to* U is the isometry $\sigma\in O(V)$ given by $\sigma|U=1$, $\sigma|U^*=-1$.

Especially important are the symmetries with respect to a (non-isotropic) hyperplane, i.e., a subspace of codimension 1. Generalizing a result of E. Cartan and Dieudonné we have the

THEOREM 2. *Each element $\sigma\in O(V)$ is the product of at most $2n-2$ symmetries with respect to a hyperplane, where $n=\dim V$. If and only if σ is in $SO(V)$, the number of symmetries representing σ will be even.*

REMARK. If L is field, each $\sigma\in O(V)$ can be written as a product of $\leqq n$ symmetries with respect to a hyperplane (cf. Dieudonné [3]). We have not been able to prove this for a general local ring. As a consequence of Theorem 2 we have that the homomorphism h_J, (2), is a map onto.

A first characterization of the congruence commutator groups is given by the

THEOREM 3. *$\Omega(V, J)$ is being generated by the elements $(\tau\tau')^2$, where τ and τ' are symmetries with respect to hyperplanes and $h_J(\tau\tau')=1$.*

Among the consequences we have $\Omega(V)$ contains the square of each element of $SO(V)$. $SO(V)/\Omega(V)$ is commutative and each element has order $\leqq 2$. The centralizer of $\Omega(V)$ is equal to the centralizer of $SO(V)$ in $O(V)$ and consists of the elements 1 and -1. center $SO(V)$ $=SO(V)\cap$center $O(V)$. center $\Omega(V)=\Omega(V)\cap$center $O(V)$.

3. The Clifford algebra and the spinor norm. A second characterization of the congruence commutator group. The *Clifford algebra over* V, $C(V)$, is defined in the usual way, cf. Bourbaki [2]. Denote by $C^+(V)$ the subalgebra of $C(V)$ generated by the products of an even number of vectors. In the multiplicative group of $C^+(V)$ we have the *special Clifford group*, $D(V)$, consisting of the products of an even number of nonisotropic vectors, considered as elements of $C(V)$. $D(V)/L^*$ is canonically isomorphic to $SO(V)$.

On $D(V)$ we have a canonical homomorphism N into the multiplicative group L^* (Bourbaki [2] calls this homomorphism spinor norm) which is quadratic in L^*. Since $SO(V)$ is isomorphic to $D(V)/L^*$, N determines a homomorphism

$$\theta: SO(V) \to L^*/L^{*2} \tag{3}$$

which we call *spinor norm.*

For $\sigma \in SO(V)$, $\theta(\sigma)$ is determined as follows: According to Theorem 3, σ can be represented as the product of an even number of symmetries τ_i with respect to nonisotropic hyperplanes H_i. For each i, choose a nonisotropic vector $A_i \in H_i^\bullet$. Then $\theta(\sigma) = \prod \Phi(A_i, A_i)L^{*2}$.

kernel(θ) is called *reduced orthogonal group over* V, $O'(V)$. Obviously $\Omega(V) \subset O'(V)$.

If ind $V \geqq 1$, then $O'(V) = \Omega(V)$ and the spinor norm θ, (3), plays for the orthogonal groups the same role which the determinant plays for the linear groups. In particular, the spinor norm yields a second characterization of the congruence commutator groups:

THEOREM 4. *Assume* ind $V \geqq 1$. *Let J be an ideal of L with $J \subset I$. Denote by θ_J the homomorphism*

$$\theta \times h_J: \sigma \in SO(V) \to (\theta(\sigma), h_J) \in L^*/L^{*2} \times SO(g_J V),$$

(i) kernel(θ: $SO(V) \to L^*/L^{*2}$) $= \Omega(V)$
kernel(θ_J: $SO(V) \to L^*/L^{*2} \times SO(g_J V)$) $= \Omega(V, J)$,

(ii) $SO(V)/\Omega(V)$ *is isomorphic to* image(θ) $= L^*/L^{*2}$, *and* $SO(V, J)/\Omega(V, J)$ *is isomorphic to* image($\theta_J | SO(V, J)$) $=$ *the subgroup of* $L^*/L^{*2} \times$center $SO(g_J V)$ *consisting of the pairs* $(\bar{a}, \bar{\sigma})$ *with* $g_J \bar{a} = \theta(\bar{\sigma})$.

(iii) $(SO(V, J) \cap \Omega(V))/\Omega(V, J)$ *is isomorphic to* center $\Omega(g_J V)$.

COROLLARY. (i) kernel(h_J: $\Omega(V) \to \Omega(g_J V)$) $= \Omega(V, J)$,

(ii) *If* dim V *odd, i.e., if* center $SO(g_J V) = 1$, *then* $SO(V, J)/\Omega(V, J)$ *is isomorphic to* $g_J^{-1}(L/J)^{*2}/L^{*2}$,

(iii) *If* dim V *odd, then* center $\Omega(g_J V) = 1$ *and therefore*: $SO(V, J) \cap \Omega(V) = \Omega(V, J)$.

4. The projective linear groups in 2 variables over a local ring. Let L be a local ring with greatest ideal $I \neq L$. Assume char$(L/I) \neq 2$ and

$L/I \neq F_3$. In [5] we have defined the *general* and the *special linear group in 2 variables over L*, denoted by $GL(2, L)$ and $SL(2, L)$, respectively.

For each ideal J of L we have the canonical homomorphism

$$h_J: GL(2, L) \to GL(2, L/J). \tag{4}$$

Here, $GL(2, L/L)$ denotes the unit group. Using the map h_J, (4), and the determinant, we have, for each ideal J of L, the following two invariant subgroups, cf. [5]:

$$GC(2, L, J) = \text{group of the } \sigma \in GL(2, L) \text{ with } h_J\sigma \in \text{center } GL(2, L/J),$$

$$SC(2, L, J) = \text{group of the } \sigma \in GL(2, L) \text{ with } h_J\sigma = 1 \text{ and } \det \sigma = 1.$$

NOTE. $GC(2, L, L) = GL(2, L)$; $SC(2, L, L) = SL(2, L)$.

Consider the canonical homomorphism

$$P: GL(2, L) \to GL(2, L)/\text{center } GL(2, L). \tag{5}$$

The image $PGL(2, L)$ of $GL(2, L)$ under the map (5) is called *projective linear group in 2 variables over L.*

The homomorphism h_J, (4), induces a homomorphism of $PGL(2, L)$ into $PGL(2, L/J)$ which we again denote by h_J. The determinant induces a map: $PGL(2, L) \to L^*/L^{*2}$ which we again denote by det. Then the images $PGC(2, L, J)$ and $PSC(2, L, J)$ of the groups $GC(2, L, J)$ and $SC(2, L, J)$, respectively, under the map P, (5), can be characterized as follows:

$$PGC(2, L, J) = \text{group of the } \sigma \in PGL(2, L) \text{ with } h_J\sigma = 1 \in PGL(2, L/J),$$

$$PSC(2, L, J) = \text{group of the } \sigma \in PGL(2, L) \text{ with } h_J\sigma = 1 \text{ and } \det \sigma = L^{*2}.$$

In [5] we have determined the structure of the group $GL(2, L)$. Together with the preceding remarks, this yields the following

STRUCTURE THEOREM FOR $PGL(2, L)$.

(i) *Each subgroup G of $PGL(2, L)$ which is invariant under $PSL(2, L)$ determines an ideal J of L such that*

$$PSC(2, L, J) \subset G \subset PGC(2, L, J) \tag{*}$$

and, conversely, each subgroup G of $PGL(2, L)$ satisfying (*) is invariant in $PGL(2, L)$.

(ii) *In $PSL(2, L)$ all the invariant subgroups are of the form $PSC(2, L, J)$, where J runs through the ideals of L.*

5. The isomorphisms of certain orthogonal groups over V with projective linear groups in 2 variables, for dim $V=3$ and 4. The properties of the Clifford algebra $C(V)$ over V yield in a natural way an isomorphism of certain orthogonal groups over 3- and 4-dimensional spaces into projective linear groups in 2 variables. These isomorphisms are of fundamental importance for the determination of the structure of orthogonal groups of spaces with arbitrary dimension, cf. §6.

THEOREM 5.

(i) *Assume* dim $V=3$ *and* ind $V=1$ *and* $L/I\neq F_3$. *Then* $SO(V)$ *is isomorphic to* $PGL(2, L)$. *Under this isomorphism, the group* $SO(V, J)$ *goes into* $PGC(2, L, J)$ *and the group* $\Omega(V, J)$ *goes into* $PSC(2, L, J)$. *In particular,* $\Omega(V)$ *goes into* $PSL(2, L)$.

(ii) *Assume* dim $V=4$ *and* ind $V=$ ind $g_I V=1$. *Then* $PSO(V) = SO(V)/\text{center } SO(V)$ *is isomorphic to a subgroup* $PU(2, L')$ *of* $PGL(2, L')$, *where* $L'=L(\Delta^{1/2})$ *is a local ring, obtained from* L *by a quadratic extension with an element* $\Delta\in\theta(-1)$. *Under this isomorphism, the group* $P\Omega(V, J)$, *isomorphic to* $\Omega(V, J)$, *goes into* $PSC(2, L', J')$, $J'=JL'$. *In particular,* $P\Omega(V)$, *isomorphic to* $\Omega(V)$, *goes into* $PSL(2, L')$.

6. The structure of the groups $SO(V)$ and $\Omega(V)$ for ind $V\geqq 1$. Let G be a subgroup of $SO(V)$. The *order of* G, $o(G)$, is the smallest ideal J with $SO(V, J)\supset G$, i.e., the smallest ideal J with $h_J G\subset$center $SO(g_J V)$.

Assume ind $V\geqq 1$. From Theorem 3 we see that $\Omega(V, J)$ has order J. From Theorem 4 (ii) we have that $SO(V, J)/\Omega(V, J)$ is commutative, and since $SO(V, J)$ and $\Omega(V, J)$ are invariant, we have: Each subgroup G of $SO(V)$ satisfying $\Omega(V, J)\subset G\subset SO(V, J)$ is invariant and of order J. The following theorem asserts that these are the only invariant subgroups (and even the only subgroups invariant under $\Omega(V)$) of $SO(V)$ of order J:

THEOREM 6. *Let* V *be a space over a local ring* L, char $L/I\neq 2$. *Assume* ind $V\geqq 1$, dim $V\geqq 3$. *If* dim $V=3$, *assume* $L/I\neq F_3$. *If* dim $V=4$, *assume* ind $g_I V=1$.

(i) *Each subgroup* G *of* $SO(V)$ *of order* $o(G)=J$ *which is invariant under* $\Omega(V)$ *satisfies the conditions*

$$(*) \qquad \Omega(V, J) \subset G \subset SO(V, J).$$

Conversely, every subgroup G *of* $SO(V)$ *satisfying* (*) *is invariant in* $SO(V)$ *and of order* J.

(ii) *Each invariant subgroup* G *of* $\Omega(V)$ *of order* $o(G)=J$ *is of the form* $G=\Omega(V, J)$ *or* $SO(V, J)\cap\Omega(V)$. $(SO(V, J)\cap\Omega(V))/\Omega(V, J)$ *is isomorphic to* center $\Omega(g_J V)$.

Remark. The proof of the preceding results is based upon an elaboration of the methods which have been developed, for an essential part by Dieudonné, for the investigation of the orthogonal groups over a field, cf. also Artin [1].

In particular, Theorem 6 is proved by relating the structure of the group $SO(V)$ with the structure of the groups mentioned in Theorem 5, and the structure of these latter groups is known, as we have stated in §4, due to the results of our earlier paper [5].

References

1. E. Artin, *Geometric algebra*, New York, Interscience, 1957.
2. N. Bourbaki, *Algèbre*, Paris, Hermann, 1959, Chapter 9.
3. J. Dieudonné, *Sur les groupes classiques*, Paris, Hermann, 1948.
4. ———, *La géométrie des groupes classiques*, Berlin, Springer-Verlag, 1955.
5. W. Klingenberg, *Linear groups over local rings*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 294–296.

Mathematisches Institut, Göttingen, Germany

TOPOLOGICAL DYNAMICS ON NILMANIFOLDS

L. AUSLANDER,[1] F. HAHN AND L. MARKUS[2]

Communicated by W. S. Massey, January 18, 1961

There has always been a lack of examples of compact manifolds which are minimal sets under the action of the real line, minimal meaning that each orbit is dense in the manifold. All tori admit such an action and G. A. Hedlund [4] has given examples of 3-manifolds having such an action.

The action of a group T on a metric space X is said to be distal if for any two distinct points $x, y \in X$ the distance between tx and ty bounded away from zero for $t \in T$. In the works of R. Ellis [1] and W. H. Gottschalk [2] the question of whether a distal minimal set is equicontinuous has arisen. Indeed, R. Ellis has shown that if X is locally compact and zero dimensional and if T acts on X so the action is distal and minimal then the action is equicontinuous.

The authors have shown that every compact nilmanifold M admits a flow under which M is minimal. This action is even real analytic. Thus there is no scarcity of manifolds which are minimal under a flow. We have also shown that these actions are distal. If these actions were equicontinuous it would follow that M would be a torus [3]. Since there are nilmanifolds which are not tori, we have shown the existence of analytic flows on compact manifolds which are distal, minimal, but not equicontinuous.

Our basic approach is as follows:

LEMMA. *If W is an open subset of a connected, simply connected, nilpotent Lie group N with discrete uniform subgroup D, then the set swept out by the one parameter subgroups containing some element of W contains a fundamental domain for N/D.*

THEOREM. *If D is a discrete uniform subgroup of a connected, simply connected, nilpotent Lie group N then the set of points of N, which lie on one parameter subgroups whose projection on N/D is dense, is a set of category* II.

By a relatively straightforward calculation we can show that the action of any one parameter subgroup on N/D is distal. From Ellis

[1] Research supported by N. S. F. Grant 15565 and O. O. R. contract SAR-DA-19-020 ORD-5254.

[2] Research supported by N. S. F. Grant 11287.

[1] we can conclude that the action of each one parameter subgroup is pointwise almost periodic. The theorem then shows that there exist one parameter subgroups such that N/D is an orbit closure. Since the action is pointwise almost periodic it follows that N/D is minimal [3].

Complete proofs of these results will be published elsewhere.

References

1. R. Ellis, *Distal transformation groups*, Pacific J. Math., vol. 8 (1958) pp. 401–405.

2. W. H. Gottschalk, *Minimal sets: An introduction to topological dynamics*, Bull. Amer. Math. Soc. vol. 64 (1958) pp. 336–351.

3. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloquium Publications, vol. 36, 1955.

4. G. A. Hedlund, *Fuchsian groups and transitive horocycles*, Duke J. Math. vol. 2 (1936) pp. 530–542.

Indiana University,
Yale University and
University of Minnesota

REPORT: AN EXAMINATION OF A DECISION PROCEDURE

BY F. C. OGLESBY

Communicated by Everett Pitcher, February 5, 1961

1. Introduction. R. Stanley in [5] presents a proof procedure for the universal validity (validity in all nonempty domains) of formulas of the first order predicate calculus which is relatively easy to apply. He shows that his procedure is a decision procedure for the monadic predicate calculus. Further, he states that the limits of the procedure, short of being a general method of decision, are not known, but that every universally valid formula which he has tested has been shown by his procedure to be valid.

W. Ackermann, in his review of Stanley's paper [1], gives an example of a universally valid formula of degree higher than two (see §3 below) for which Stanley's procedure gives no decision and also suggests a way to enlarge the realm of application of the procedure; namely, to start not with a given formula A, but with a finite disjunction $A \vee \cdots \vee A$.

The purpose of this note is to report on an examination of Stanley's procedure to determine for certain decidable classes of formulas whether or not the procedure gives a method of decision. We shall use SP to denote Stanley's procedure.

Throughout this note, the familiar propositional connectives are denoted by '$\wedge$' (and), '$\vee$' (or), and '$\neg$' (not). Individual variables are denoted by x, x_1, $x_2, \cdots, y, y_1, y_2, \cdots$, and predicate variables by $F_1^1, F_2^1, \cdots, F_1^2, F_2^2, \cdots, G_1^2, G_2^2, \cdots, H_1^2, H_2^2, \cdots$, the superscript indicating the monadic or dyadic character of the variable. We shall feel free to omit subscripts and superscripts whenever no ambiguity will arise. Universal quantifiers are denoted by $(\forall x)$, $(\forall x_1), \cdots, (\forall y), (\forall y_1), \cdots$. We assume throughout that existential quantifiers and signs of material implication or equivalence have been replaced by their respective usual equivalents. $A, A_1, \cdots, B, B_1, \cdots$ denote arbitrary formulas; $A(x), \cdots, B(x), \cdots$ denote arbitrary formulas with free x; etc.

The general method of SP is to derive a contradiction from the negation of the given formula. In particular, a preliminary step of exportation (as characterized in W. Quine [6]) is applied to the universal closure of the given formula, followed by a preliminary step of prefixing a '$\neg$'. The remainder of the procedure, which we will denote by SP*, involves cycles of prescribed steps including instantiations of quantifications according to natural deduction techniques and applications of propositional rules. Contradictory disjuncts are dropped as they are uncovered and, if at some step everything vanishes, then the original formula is established as universally valid, or, equivalently, the formula resulting from the preliminary steps is established as nonsatisfiable (satisfiable in no nonempty domain). In this case, we say that SP yields a contradiction.

2. We say that a formula A is a member of the AE predicate calculus if the formula resulting from A by applying the preliminary steps of SP contains no negative quantifier within the scope of a positive quantifier. (Positive and negative quantifiers are defined as with J. Herbrand [2]; when converted to prenex normal form, positive quantifiers appear in the prefix as universal quantifiers and negative as existential.) Note that by this definition the monadic predicate calculus is contained in the AE predicate calculus. Thus, the following theorem is an extension of Stanley's result that the procedure gives a method of decision for the monadic predicate calculus.

Theorem 1. *SP gives a method of decision for the AE predicate calculus.*

Proof. Let A be an arbitrary universally valid formula of the AE predicate calculus. Then A^*, the result of applying the preliminary steps of SP to A, is nonsatisfiable and has an equivalent prenex normal form:

$$\neg(\forall x_1) \neg \cdots \neg (\forall x_n) \neg (\forall y_1) \cdots (\forall y_m) B(x_1, \cdots, x_n, y_1, \cdots y_m), \quad n \geqq 0.$$

Hence, as is well-known, A^* is not satisfiable in any domain containing n individuals if $n>0$, and A^* is not satisfiable in any domain containing one individual if $n=0$. Thus, upon applying to A^* the procedure in Hilbert and Ackermann [3] for determining satisfiability in finite domains, a truth-functional contradiction is obtained. It follows easily from this that SP* applied to A^* yields a contradiction.

3. A formula is of second degree if at least one quantifier appears within the scope of another quantifier, but no quantifier appears within the scope of more than one other quantifier. K. J. Hintikka in [4] extends the notion of distributive normal form in the propositional calculus to the full predicate calculus. In particular, he defines closed second degree distributive normal forms. These latter normal forms are finite disjunctions of formulas called closed second degree

constituents of zero order. Following Hintikka, we denote an arbitrary closed second degree constituent of zero order by $C^0 2$. Hintikka presents a set of three conditions with the property that an arbitrary $C^0 2$ is nonsatisfiable if and only if at least one of the conditions holds for $C^0 2$. However, a proof of the necessity of these conditions is omitted in his monograph. We obtain a proof of the necessity of these conditions by showing that if none of the conditions holds for an arbitrary $C^0 2$, then $C^0 2$ is satisfiable in a denumerably infinite domain.

4. THEOREM 2. *SP* gives a method of decision for nonsatisfiability in the class of all closed second degree distributive normal forms.*

PROOF. The theorem follows by showing that SP* yields a contradiction when applied to an arbitrary $C^0 2$ satisfying at least one of the above three conditions. It should be mentioned that there are certain special cases for which the set of necessary and sufficient conditions does not apply, but that these cases are easily handled.

5. We say that a formula A of the predicate calculus is a member of the class Φ if the formula A^* which results from applying the preliminary steps of SP to A is (closed and) of second degree. Note that if A^* is of first degree (no quantifier appears within the scope of another quantifier), then A is a member of the AE predicate calculus and Theorem 1 applies.

Since any arbitrary closed second degree formula A^* can be effectively transformed into an equivalent closed second degree distributive normal form, the conditions of §3 and Theorem 2 each give a method of decision for determining universal validity in the class Φ. However, it is not, in general, practical to attempt the transformation, and hence it is of interest to determine whether SP necessarily yields a contradiction when applied directly to an arbitrary universally valid formula of Φ. The following theorem gives a negative answer to this question.

THEOREM 3. *SP is not a decision procedure for determining universal validity of arbitrary formulas of Φ.*

PROOF. Let $A(x)$ denote

$$[Fx \wedge \neg (\forall y) \neg (Gxy \wedge \neg Fy)] \vee [Fx \wedge \neg (\forall y) \neg (Hxy \wedge \neg Fy)].$$

Consider $B_1 = \neg (\forall x) A(x)$. The universal validity of B_1 is easily determined by a direct valuation. However, it can be shown that SP applied to B_1 does not yield a contradiction.

REMARK. We note that it can be shown that SP does yield a contradiction when applied to the formula

$$\neg(\forall x)\{Fx \wedge [\neg(\forall y)\neg(Gxy \wedge \neg Fy) \vee \neg(\forall y)\neg(Hxy \wedge \neg Fy)]\},$$

which is equivalent to B_1. However, a simple application of the distributive law does not, in general, remedy the situation; for the following formula, B_2, is universally valid but does not yield a contradiction under application of SP:

$$\neg(\forall x)\{[\neg(\forall y)\neg(Fxy \wedge \neg Fyy) \wedge Fxx] \vee [\neg(\forall y)\neg(Fyx \wedge \neg Fyy) \\ \wedge (\forall y)((\neg Fxy \wedge \neg Fyy) \vee (Fyx \wedge Fyy)) \wedge \neg Fxx]\}.$$

6. Ackermann's suggestion to start not with a given formula A, but rather with a finite disjunction $A \vee \cdots \vee A$ does strengthen SP for the class Φ, for it can be shown that SP applied to $B_1 \vee B_1$ yields a contradiction. Moreover, it can be shown that although SP applied to $B_2 \vee B_2$ does not yield a contradiction, SP applied to $B_2 \vee B_2 \vee B_2$ does yield a contradiction (these results also establishing the universal validity of B_1 and B_2).

It is clear, however, that the suggestion cannot give us a method of decision for the class Φ unless we can give some sort of a rule for determining the number of disjuncts to be used. For convenience, let us denote SP modified by starting with q disjuncts of the given formula by SP_q.

THEOREM 4. *No positive integer M exists such that SP_M is a decision procedure for determining universal validity of arbitrary formulas of Φ.*

PROOF. Let m be an arbitrary positive integer. Consider the formula

$$B'_m = \neg(\forall x)A_1(x) \wedge \neg(\forall x)A_2(x) \wedge \cdots \wedge \neg(\forall x)A_m(x),$$

where

$$A_k(x) = [F_k x \wedge \neg(\forall y)\neg(G_k xy \wedge \neg F_k y)] \\ \vee [F_k x \wedge \neg(\forall y)\neg(H_k xy \wedge \neg F_k y)], \quad 1 \leqq k \leqq m.$$

The universal validity of B'_m is easily established by a direct valuation. However, it can be shown that, if $q \leqq m$, then SP_q applied to B'_m does not yield a contradiction, whereas SP_{m+1} applied to B'_m does yield a contradiction (also establishing the universal validity of B'_m).

REMARK. Theorem 4 does not exclude the possibility of being able to give, for the class Φ, an effective rule which, when applied to a given individual formula A of Φ, will produce an integer N (depending upon A) such that: if A is universally valid, then SP_N applied to A yields a contradiction.

REFERENCES

1. W. Ackermann, A review of [5], J. Symb. Logic vol. 21 (1956) p. 197.

2. J. Herbrand, *Recherches sur la théorie de la démonstration*, Trav. Soc. Sci. Lett. Varsovie, Classe III, no. 33, 1930.

3. D. Hilbert and W. Ackermann, *Principles of mathematical logic*, New York, Chelsea Publishing Company, 1950.

4. K. J. Hintikka, *Distributive normal forms in the calculus of predicates*, Acta Philosophica Fennica vol. 6 (1953) 71 pp.

5. R. Stanley, *An extended procedure in quantificational logic*, J. Symb. Logic vol. 18 (1953) pp. 97–104.

6. W. V. Quine, *On the logic of quantification*, J. Symb. Logic vol. 10 (1945) pp. 1–12.

LEHIGH UNIVERSITY

ON THE SEMIGROUP OF IDEAL CLASSES IN AN ORDER OF AN ALGEBRAIC NUMBER FIELD

BY E. C. DADE, O. TAUSSKY AND H. ZASSENHAUS

Communicated by N. Jacobson, December 30, 1960

There is a natural link between classes of ideals in orders of algebraic number fields and similarity classes of integral matrices defined by unimodular matrices.

Two fractional ideals in an order of an algebraic number field are called *arithmetically equivalent* if and only if they differ by a factor in the field. It is known that the number of classes obtained in this way is finite and that the classes form a finite abelian semigroup. In order to study and generalize these ideal classes orders in finite extensions of more general fields are considered.

In order to describe the results obtained several abstract concepts concerning semigroups are introduced:

An element a of a multiplicative semigroup S is called *invertible* if the equations

$$ax = ya = e, \qquad ea = ae = a$$

hold for some elements x, y, e of S.

It follows then that e is uniquely determined, as a function of a, that it is an idempotent and that a has a unique inverse with respect to e, namely ex. All invertible elements with the same identity e form a multiplicative group $G(e)$. A semigroup is called *pure* if every invertible element is an idempotent.

Two elements a, b of S are called *weakly equivalent* if the equations

$$ax = b, \qquad by = a$$

can be solved in S.

Consequently weak equivalence classes can be introduced. Every idempotent weak equivalence class contains exactly one idempotent.

These concepts are now applied to the abelian semigroup of classes of ideals in an algebraic extension E of a field F. Such an ideal is defined as an $\mathfrak{o}_F$-module, formed of elements in E, where $\mathfrak{o}_F$ is a Dedekind ring in F with F as quotient field. The ideals are assumed finitely generated over $\mathfrak{o}_F$ and to contain a basis of E over F.

This set of ideals $\mathfrak{a}$ is closed with respect to multiplication, addition, intersection, quotient $\mathfrak{a}:\mathfrak{b}$ (i.e. the set of elements x in E satisfying $x\mathfrak{b}\subseteq\mathfrak{a}$) and the adjoint operation $\mathfrak{a}^T$ (the set of elements x of E satisfying the condition that

$$\operatorname{tr} xy \in \mathfrak{o}_F$$

for all y of $\mathfrak{a}$). For every ideal $\mathfrak{a}$ we denote the order $\mathfrak{a}:\mathfrak{a}$ by $\mathfrak{o}_{\mathfrak{a}}$.

In the special case where F is the rational number field and $\mathfrak{o}_F$ the ring of rational integers the ideal classes form a finite commutative semigroup under arithmetical equivalence.

To study these ideals the following lemma of Krull is used:

Let $\mathfrak{a}$, $\mathfrak{b}$ *be two ideals which satisfy the relations*

$$\mathfrak{a}\mathfrak{b} = \mathfrak{b}, \qquad \mathfrak{a}\mathfrak{a} \subseteq \mathfrak{a}.$$

Then $\mathfrak{a}$ *is an order, i.e., an ideal which contains a unit element and is closed under multiplication.*

The lemma implies that every idempotent ideal is an order. It further implies that two ideals $\mathfrak{a}$, $\mathfrak{b}$ are weakly equivalent if and only if

$$1 \in (\mathfrak{b}:\mathfrak{a})(\mathfrak{a}:\mathfrak{b}).$$

If F is a finite extension of the rational field and $\mathfrak{o}_F$ the ring of algebraic integers then a power of every ideal of E is an invertible ideal. For this same special case we have:

Every idempotent weak equivalence class is represented by precisely one order. Conversely, for a given order $\mathfrak{o}$ of E over $\mathfrak{o}_F$ all $\mathfrak{o}$-ideals $\mathfrak{a}$ satisfying

$$\mathfrak{a}(\mathfrak{o}:\mathfrak{a}) = \mathfrak{o}$$

form a multiplicative group $G(\mathfrak{o})$ with $\mathfrak{o}$ as identity. This group is the idempotent weak equivalence class represented by $\mathfrak{o}$.

In the general case we have: All invertible ideals $\mathfrak{x}$ of E over $\mathfrak{o}_F$ with the same identity order $\mathfrak{o}_{\mathfrak{x}}$ form a multiplicative abelian group $G(\mathfrak{o})$.

For two orders $\mathfrak{o}$, $\mathfrak{o}'$ of E over $\mathfrak{o}_F$ satisfying $\mathfrak{o} \subseteq \mathfrak{o}'$ and $\mathfrak{o}:\mathfrak{o}' \neq (0)$ the mapping $\sigma(\mathfrak{o}, \mathfrak{o}')$ given by

$$\sigma(\mathfrak{o}, \mathfrak{o}')\mathfrak{x} = \mathfrak{x}\mathfrak{o}' \qquad (\mathfrak{x} \in G(\mathfrak{o}))$$

is a homomorphism of $G(\mathfrak{o})$ onto $G(\mathfrak{o}')$.

Take again the case when F is a finite extension of the rational field and $\mathfrak{o}_F$ the ring of integers in it. It is concluded that every ideal $\mathfrak{a}$ in E over $\mathfrak{o}_F$ is weakly equivalent to an ideal $\mathfrak{b}$ such that $\mathfrak{b}^\rho$ is an order $\mathfrak{o}_1$, for some integer ρ and such that

$$\mathfrak{b} \subseteq \mathfrak{o}_1,$$
$$\mathfrak{b}\mathfrak{o}_1 = \mathfrak{o}_1.$$

For the proof the fact that some power of $\mathfrak{a}$ is invertible is used. This follows by finiteness considerations.

It is shown next that for the general case the following fact holds:[1] *Let $E/F=n$ and let $\mathfrak{a}^\rho$ be the least positive invertible power of the ideal $\mathfrak{a}$ in E, then*

$$\rho \leqq n-1.$$

Further $n-1$ is best possible.

For the proof of this the following general lemma is proved:

Let H be a commutative hypercomplex system with unit element of dimension n over the field F. If for a linear subspace M of H there is a positive integer r such that

$$M^r = H$$

then also

$$M^{n-1} = H.$$

The following generalization can also be proved: *if H has a faithful representation by $\mu\times\mu$ matrices then*

$$M^{\mu-1} = H.$$

Finally, a "reduction" theorem is proved. First, observe that in any commutative ring R with unit element and a subring $\mathfrak{o}$ containing the unit element the $\mathfrak{o}$-modules contained in R form a system $I^*(R/\mathfrak{o})$ that is closed under addition, multiplication, intersection and quotient forming.

Furthermore let us introduce the relation of weak equivalence between two $\mathfrak{o}$-modules $\mathfrak{a}$, $\mathfrak{b}$ contained in R:

"$\mathfrak{a}$ is weakly equivalent to $\mathfrak{b}$" if and only if

$$\mathfrak{a}(\mathfrak{b}:\mathfrak{a}) = \mathfrak{b}, \qquad \mathfrak{b}(\mathfrak{a}:\mathfrak{b}) = \mathfrak{a}.$$

This relation is reflexive, symmetric, transitive and multiplicative so that the weak equivalence classes form an abelian semigroup $W^*(R/\mathfrak{o})$. Among these the classes containing a representative $\mathfrak{a}$ satisfying

$$\mathfrak{o}_\mathfrak{a} \supseteq \mathfrak{o}, \qquad \mathfrak{a}^q = \mathfrak{o}_{\mathfrak{a}^q}$$

for some exponent q, form a subsemigroup $U^*(R/\mathfrak{o})$.

---

[1] This was conjectured on the basis of an example of an order in the cubic field generated by $\alpha^3+2\alpha^2+2\alpha+1=0$, computed on the IBM 709 at the Western Data Processing Center by E. C. Dade and H. Zassenhaus. (An account of this computation which was sponsored in part by ONR will be published separately.) In the case of an order in a quadratic extension of the rational number field every ideal class has an inverse, for the semigroup of ideal classes is in this case a union of groups. This had already been discovered by Gauss in terms of quadratic forms.

THEOREM. *Let F be a finite extension of the rational field. Let $\mathfrak{o}_F$ be the subring of the algebraic integers of F. Let E be an extension of F of finite degree n. Let $\mathfrak{o}$, $\mathfrak{o}'$ be two orders of E over $\mathfrak{o}_F$ satisfying $\mathfrak{o}' \subseteq \mathfrak{o}$. Let*

$$\mathfrak{f} = \mathfrak{o}' : \mathfrak{o}.$$

The weak equivalence classes of ideals $\mathfrak{a}$ of E over $\mathfrak{o}_F$ satisfying

(1) $\qquad \mathfrak{o}_{\mathfrak{a}} \supseteq \mathfrak{o}',$ $\qquad\qquad$ (2) $\qquad \mathfrak{o}_{\mathfrak{a}^{n-1}} \subseteq \mathfrak{o}$

form an abelian semigroup $W(\mathfrak{o}, \mathfrak{o}'/\mathfrak{o}_F)$ that is isomorphic to $U^*(\mathfrak{o}/\mathfrak{f})/(\mathfrak{o}'/\mathfrak{f})$.

CALIFORNIA INSTITUTE OF TECHNOLOGY AND
UNIVERSITY OF NOTRE DAME

AREA OF DISCONTINUOUS SURFACES

BY CASPER GOFFMAN[1]

Communicated by J. W. T. Youngs, January 19, 1961

1. A general theory of surface area, [1; 2], exists for the non-parametric case. Thus, area is defined for all measurable f on the unit square $Q=I\times J$. The area functional is lower semi-continuous with respect to almost everywhere convergence and agrees with the Lebesgue area for continuous f. On the other hand, for continuous parametric mappings T of the closed unit square Q into euclidean 3-space E_3, Lebesgue area is not lower semi-continuous with respect to almost everywhere convergence nor even, as C. J. Neugebauer has shown, [3], with respect to pointwise convergence.

It thus appears that a theory of parametric surface area must be restricted to surfaces which cannot deviate too far from the ones given by continuous mappings. In this paper, we develop the beginnings of a theory for a class of surfaces which we call linearly continuous.

2. Let f be a real function defined on Q and, for every u, let f_u be defined by $f_u(v)=f(u, v)$ and let f_v be defined similarly. Then f is linearly continuous if f_u is continuous for almost all u and f_v is continuous for almost all v. A mapping $T: x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$ of Q into E_3 is linearly continuous if x, y, z are linearly continuous.

A sequence $\{f_n\}$ of functions converges linearly to a function f if $(f_n)_u$ converges uniformly to f_u for almost all u, and $(f_n)_v$ converges uniformly to f_v for almost all v. A sequence $T_n: x=x_n(u, v)$, $y=y_n(u, v)$, $z=z_n(u, v)$ converges linearly to a mapping $T: x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$ if $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge linearly to x, y, z, respectively.

Let P be the set of quasi linear mappings from Q into E_3. For $p, q\in Q$ let

$$\begin{aligned} d(p, q) = \inf[k&: \text{there are sets } A_k \subset I,\ B_k \subset J,\\ &m(A_k) > 1-k,\ m(B_k) > 1-k, \text{ and } |p(u,v)-q(u,v)| < k\\ &\text{on } (A_k \times J) \cup (I \times B_k)]. \end{aligned}$$

It is easy to verify that P is a metric space and that $\{p_n\}$ converges to p in this space if and only if it converges linearly. Let E be the elementary area functional on P. It is not hard to prove

[1] Research supported by National Science Foundation Grant No. NSF G-5867.

THEOREM 1. *E is lower semi-continuous on P. In other words, if $\{p_n\}$ converges linearly to p then* $\liminf E(p_n) \geqq E(p)$.

By the Fréchet extension theorem, E is extended to a lower semi-continuous functional Φ on the completion $\mathfrak{L}$ of P.

THEOREM 2. *The completion $\mathfrak{L}$ of P is the space of linearly continuous mappings with the metric corresponding (as above) to linear convergence.*

3. It is obvious that for every continuous mapping T, $A(T) \geqq \Phi(T)$ where $A(T)$ is the Lebesgue area. The inverse inequality holds so that the functional Φ constitutes a legitimate extension of Lebesgue area to substantially wider class of mappings than the continuous ones. We outline the proof.

For a continuous T: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, the lower area $V(T)$ is defined as follows:

Let T_1: $y=y(u, v)$, $z=z(u, v)$, T_2: $x=x(u, v)$, $z=z(u, v)$, and T_3: $x=x(u, v)$, $y=y(u, v)$ be the associated flat mappings. For every simple polygonal region P in Q^0, let

$$v_1(P) = \int |O(\xi, T_1P^*)|,$$

where the integration is over the yz plane, and $O(\xi, T_1P^*)$ is the topological index of T_1P^* at ξ (A^0 and A^* are the interior and boundary, respectively, of a set A). Define $v_2(P)$ and $v_3(P)$, similarly, and let

$$v(P) = [v_1(P)^2 + v_2(P)^2 + v_3(P)^2]^{1/2}.$$

Let $\pi = (P_1, \cdots, P_n)$ be a finite set of pair-wise disjoint simple polygonal regions in Q^0 and

$$v(\pi) = \sum_{i=1}^{n} v(P_k).$$

Finally, let

$$V(T) = \sup[v(\pi): \pi].$$

Cesari has shown (e.g. [**4**]) that $A(T) = V(T)$ for every continuous T.

The distance between 2 sets A and B is defined by

$$d(A, B) = \sup[d(x, B): x \in A] + \sup[d(y, A): y \in B].$$

With this metric, the set α of simple polygonal regions is a separable metric space. Let $\beta \subset \alpha$ be dense in α and

$$V_\beta = \sup[v(\pi): \pi \subset \beta].$$

LEMMA 1. $V_\beta(T) = V(T)$.

Now, let $\{T_n\}$ be a sequence of continuous mappings which converges linearly to a continuous mapping T. Let γ be the set of simple polygonal regions whose boundaries consist of line segments parallel to the coordinate axes for which T and T_n, $n=1, 2, \cdots$ are continuous and on each of which $\{T_n\}$ converges uniformly to T. For each $\pi \subset \gamma$, lim inf $v(\pi, T_n) \geqq v(\pi, T)$. Since γ is dense in α, it follows that lim inf $V(T_n) \geqq V(T)$. This proves

THEOREM 3. *$A(T)$ is lower semi-continuous with respect to linear convergence on the set of continuous mappings.*

COROLLARY 1. *$A(T) = \Phi(T)$ for every continuous T.*

PROOF. For every sequence $\{p_n\}$ of quasi-linear mappings converging linearly to T, lim inf $E(P_n) \geqq A(T)$. Choose $\{p_n\}$ so that lim $E(p_n) = \Phi(T)$. Then $A(T) \leqq \Phi(T)$,

4. A set S will be called negligible if $S \subset Z_1 \times Z_2$ where Z_1 and Z_2 have linear measure zero. Kolmogoroff's principle holds in the following form.

THEOREM 4. *If T_1 and T_2 are linearly continuous mappings from Q into E_3 and if for every pair of points ξ, η not belonging to a negligible set*

$$| T_1\xi - T_1\eta | \leqq | T_2\xi - T_2\eta |,$$

then $\Phi(T_1) \leqq \Phi(T_2)$.

5. A real function f on Q is BVC if for almost all u and almost all v, f_u and f_v are equivalent to functions of bounded variation and the corresponding variation functions are summable. f is ACE if for almost all u and almost all v, f_u and f_v are equivalent to absolutely continuous functions.

For functions which are BVT and ACT it is a simple known fact that the integral means commute with the partial derivatives. This also holds almost everywhere for functions which are BVC and ACE. Using this fact and the fact, [**5**], that if f is BVC and linearly continuous then the integral means of f converge linearly to f, the proof of the following generalization of a theorem of Morrey, [**4**], may be obtained in somewhat standard fashion. The generalization is in two directions. Instead of holding only for conjugate Lebesgue spaces, the theorem holds for conjugate Köthe spaces, [**6**; **7**], and the theorem holds for linearly continuous mappings rather than just for continuous ones.

THEOREM 5. *If the functions x, y, z of a linearly continuous T are BVC and ACE and if the pairs of partial derivatives (x_u, y_v), (x_v, y_u), (x_u, z_v), (x_v, z_u), (y_u, z_v), (y_v, z_u) belong to conjugate Köthe spaces, the area $\Phi(T)$ is given by the formula*

$$\Phi(T) = \int J\, dudv$$

where $J=[J_1^2+J_2^2+J_3^2]^{1/2}$ and J_1, J_2, J_3 are the jacobians of T_1, T_2, T_3, respectively.

6. We define an equivalence relation for linearly continuous mappings. T is equivalent to $T'(T\approx T')$ if there are sequences $\{p_n\}$ and $\{q_n\}$ of quasi linear mappings such that, for every n, $p_n\approx q_n$ in the Lebesgue sense and $\{p_n\}$ converges linearly to T, $\{q_n\}$ converges linearly to T'.

The following simple facts hold:

(a) The relation "$\approx$" has the properties of an equivalence relation.

(b) If T and T' are continuous and Fréchet equivalent then $T\approx T'$.

(c) If $T\approx T'$ then $\Phi(T)=\Phi(T')$.

We refer to an equivalence class as a surface and to its elements as representations.

D mappings, the Dirichlet integral, and almost conformal mappings are defined as for the continuous case, [4], with BVT and ACT replaced by BVC and ACE.

We say that a mapping T is simple if there is a negligible set S such that $\xi\in Q-S$, $\eta\in Q-S$, $\xi\neq\eta$ implies $T(\xi)\neq T(\eta)$.

The following holds:

THEOREM 6. *If T' is a linearly continuous simple mapping and $\Phi(T')<\infty$, the surface given by T' has a representation T, with jacobian J, such that*

$$\Phi(T') = \Phi(T) = \int J\, dudv.$$

COROLLARY. *Every linearly continuous nonparametric surface of finite area has a parametric representation T, with jacobian J, such that*

$$\Phi(T) = \int J\, dudv$$

Bibliography

1. L. Cesari, *Sulle funzioni a variazione limitata*, Ann. Scuola Norm. Sup. Pisa vol. 5 (1936) pp. 299–313.

2. C. Goffman, *Lower semi-continuity and area functionals*. I, *The non-parametric case*, Rend. Circ. Mat. Palermo vol. 2 (1954) pp. 203–235.

3. C. J. Neugebauer, *Lebesgue area and pointwise convergence*, Abstract 554-13, Notices Amer. Math. Soc. vol. 6 (1959) p. 80.

4. L. Cesari, *Surface area*, Princeton, University Press, 1956.

5. C. Goffman, *Area of linearly continuous functions*, Acta Math. vol. 103 (1960) pp. 269–290.

6. J. Dieudonné, *Sur les espaces de Köthe*, J. Anal. Math. vol. 1 (1951) pp. 81–115.

7. C. Goffman, *Completeness in topological vector lattices*, Amer. Math. Monthly vol. 66 (1959) pp. 87–92.

Purdue University

ON THE PRIME IDEALS OF SMALLEST NORM IN AN IDEAL CLASS mod $\mathfrak{f}$ OF AN ALGEBRAIC NUMBER FIELD

BY G. J. RIEGER

Communicated by I. J. Schoenberg, February 3, 1961

In 1947, Linnik [3] proved the following theorem:

THEOREM (OF LINNIK). *There exists an absolute constant c such that in every prime residue class mod k there is a prime number p with* $p<k^c$.

A simplified proof of this theorem was given by Rodosskii [7] whose proof (similar to Linnik's) rests basically on (A) function-theoretic lemmas, (B) theorems on L-functions, (C) estimates of character sums, and (D) a sieve method. The theorems (B) can be classified and characterized as follows:

(B1) order of magnitude of the L-functions [5, Chapter 4, Satz 5.4],

(B2) existence of at most one exceptional zero [5, Chapter 4, Satz 6.9],

(B3) Siegel's theorem on the exceptional zero [5, Chapter 4, Satz 8.1],

(B4) functional equation of the L-functions [5, Chapter 7, Satz 1.1],

(B5) number of zeros in vertical strips [5, Chapter 7, Satz 3.3],

(B6) explicit formulae [5, Chapter 7, Satz 4.1, Satz 6.1].

Recently, I have been able to prove the following generalization of Linnik's theorem which I had conjectured elsewhere [6, p. 168]:

THEOREM 1. *For every algebraic number field K there exists a constant* $c(K)$, *depending on K only, such that in every ideal class* mod $\mathfrak{f}$ (*in the narrowest sense*) *there is a prime ideal* $\mathfrak{p}$ *with* $N\mathfrak{p}<N\mathfrak{f}^{c(K)}$.

The skeleton of the proof of Theorem 1 can be taken from Rodosskii's proof; the lemmas (A) are the same; the generalized theorems (B1) resp. (B3) resp. (B4) resp. (B5) resp. (C) resp. (D) can be found in [1] and [4] resp. [4] resp. [1] resp. [1] resp. [2] resp. [6]; the remaining theorems (B2) and (B6) can easily be generalized. The details of the proof of Theorem 1 are then essentially the same as in [7]. This completes the outline of the proof of Theorem 1.

To the related question of the "smallest" prime *numbers* in a *residue* class mod $\mathfrak{f}$ we hope to come back soon.

BIBLIOGRAPHY

1. E. Landau, *Ueber Ideale und Primideale in Idealklassen*, Math. Z. vol. 2 (1918) pp. 52–154.

2. ———, *Verallgemeinerung eines Pólyaschen Satzes auf algebraische Zahlkoerper*, Nachr. Akad. Wiss. Goettingen (1918) pp. 478–488.

3. J. V. Linnik, *Ueber die kleinste Primzahl in einer arithmetischen Progression*, Mat. Sb. vol. 15 (1947) pp. 139–368.

4. T. Mitsui, *Generalized prime number theorem*, Jap. J. Math. vol. 26 (1956) pp. 1–42.

5. K. Prachar, *Primzahlverteilung*, Berlin, Springer-Verlag, 1957.

6. G. J. Rieger, *Verallgemeinerung der Siebmethode von A. Selberg auf algebraische Zahlkoerper*. II, J. Reine Angew. Math. vol. 201 (1959) pp. 157–171.

7. K. A. Rodosskii, *Ueber die kleinste Primzahl in einer arithmetischen Progression*, Mat. Sb. vol. 34 (1954) pp. 331–356.

PURDUE UNIVERSITY

DOCTORATES CONFERRED IN 1960

The following are among those who received doctorates in the mathematical sciences and related subjects from universities in the United States and Canada during 1960. In each case when available, the university, the month in which the degree was conferred, minor subjects (other than mathematics), and title of dissertation are given. 309 names are listed.

R. H. Abraham, University of Michigan, June, *Discontinuities in general relativity.*

R. D. Adams, University of Minnesota, June, minor in Physics, *I_p density of solutions to parabolic and related equations on space time surfaces.*

Sidney Addelman, Iowa State University, November, *Fractional factorial plans.*

D. E. Amos, Oregon State College, June, minor in Chemical Engineering, *Application of the Wiener-Hopf technique to half plane diffraction of cylindrical waves.*

K. W. Anderson, University of Illinois, June, *Midpoint local uniform convexity, and other geometric properties of Banach spaces.*

M. A. Arkowitz, Cornell University, June, *The generalized Whitehead product.*

Michael Artin, Harvard University, June, *On Enriques' surfaces.*

H. R. Axelrod, New York University, June, minor in Biology, *Mathematical basis for solution of medical and dental biostatistical problems.*

A. E. Babbitt, Jr., Columbia University, June, *Finitely generated pathological extensions of difference fields.*

O. P. Bagai, University of British Columbia, May, *Multiple comparison methods and certain distributions arising in multivariate statistical analysis.*

R. E. Barlow, Stanford University, October, *Applications of semi-Markov processes to counter and reliability problems.*

B. H. Barnes, Michigan State University, December, *Structure of automata.*

L. E. Batson, University of Texas, January, minor in Physics, *On inversion of the Laplace transformation by means of a step-function.*

Sister Marion Beiter, Catholic University of America, February, minor in Physics, *Coefficients in the cyclotomic polynomial for a number with at most three distinct odd primes in its factorization.*

Geneva Grosz Belford, University of Illinois, June, minor in Physics, *Computer logic programs.*

E. R. Berkson, University of Chicago, August, I. *Generalized diagonable operators.* II. *Some metrics on the subspaces of a Banach space.*

S. A. Bessler, Stanford University, June, *Theory and applications of the sequential design of experiments, k-actions and infinitely many experiments.*

Andrzej Bialynicki-Birula, University of California, Berkeley, June, *On automorphisms and derivations of simple rings with minimum condition.*

R. J. Bickel, University of Pittsburgh, May, *An investigation of properties of a scale of Abel type summability methods.*

L. N. Bidwell, University of Pennsylvania, June, *Regionally almost periodic transformation groups.*

J. H. Billings, University of Maryland, June, *Extensions of the Laplace cascade method.*

R. G. Bilyeu, University of Kansas, October, minor in Physics, *Perturbation of an autonomous differential equation with a parameter.*

J. J. Birch, University of California, Berkeley, September, *Approximations for the entropy for functions of Markov chains.*

F. T. Birtel, University of Notre Dame, August, *Banach algebras of multipliers.*

G. R. Blakely, University of Maryland, June, *Partitions and power series.*

T. K. Boehme, California Institute of Technology, June, *Operational calculus and the finite part of divergent integrals.*

R. A. Bonic, Yale University, June, *The involution in group algebras.*

J. R. Borsting, University of Oregon, June, *Limit theorems for censored data.*

J. J. Bowers, Carnegie Institute of Technology, June, *On symmetric means and their applications.*

R. S. Brand, Brown University, June, *The collapse of a spherical cavity in a compressible liquid.*

Mildred Jeannette Brannon, University of Illinois, February, minor in Musicology, *Rotations in locally bounded linear metric spaces which are not locally convex.*

J. D. Brooks, University of Southern California, June, minor in Physics, *Second order dissipative systems.*

R. R. Brown, University of California, Los Angeles, August, *Solution of boundary value problems using non-uniform grids.*

Judith Brostoff Bruckner, University of California, Los Angeles, June, *Triangulations of bounded distortion in the classification theory of Riemann surfaces.*

R. E. Bryan, Yale University, June, *Geodesic winding on Riemannian planes.*

J. D. Buckholtz, University of Texas, August, *Concerning polynomial sequences and the distribution of their zeros.*

Eugene Butkov, McGill University, May, *Spin-orbit potentials of nucleons.*

B. R. Buzby, Indiana University, June, *Integral equivalence of quadratic forms over local fields with* $|2| < l$.

Mary Katherine Huggin Cabell, University of Virginia, June, *Mappings with a multiplicity function.*

T. W. Cairns, Oklahoma State University, May, *A generalized derivative.*

W. V. Caldwell, University of Michigan, February, *Vector spaces of light interior orientation-preserving* C' *functions.*

D. G. Cantor, University of California, Los Angeles, August, *On sets of algebraic integers whose remaining conjugates lie in the unit circle.*

F. W. Carroll, Purdue University, January, *On some classes of noncontinuable analytic functions; difference properties for some classes of functions on locally compact groups.*

G. D. Chakerian, University of California, Berkeley, September, *Integral geometry in the Minkowski plane.*

S. U. Chase, University of Chicago, March, *Homological properties of certain rings and modules.*

S. D. Chatterji, Michigan State University, June, *Martingales of Banach-valued random variables.*

C. H. Chicks, University of Oregon, June, *Periodic automorphisms on Banach algebras.*

I. F. Christensen, Catholic University of America, June, minor in Philosophy and Psychology, *Some extensions of a theorem of Marcinkiewicz.*

F. L. Cleaver, Tulane University, August, *On coverings of four-space by spheres.*

R. F. Cogburn, University of California, Berkeley, January, *Asymptotic properties of stationary sequences.*

S. H. Coleman, University of Virginia, June, *Integration in infinite product spaces.*

F. B. Correia, University of Colorado, June, minor in Physics, *A theory of primes.*

C. G. Costley, University of Illinois, June, *Singular nonlinear integral equation with complex valued kernels of type N.*

R. C. Courter, University of Wisconsin, January, *Maximal commutative algebras of linear transformations.*

G. A. Craft, Ohio State University, August, *A transformation theory for multiplicity functions.*

R. J. Crittenden, Massachusetts Institute of Technology, February, *Conjugate and minimum points on Riemannian manifolds.*

G. J. Culler, University of California, Los Angeles, January, *Polar decomposition and boundary value problems for matrix differential equations.*

G. L. Curme, University of Illinois, February, minor in Economics, *Perron summability as related to Denjoy type quasi analytic functions.*

T. B. Curtz, Yale University, June, *A class of third order ordinary differential equations.*

E. C. Dade, Princeton University, June, *Multiplicity and monoidal transformations.*

R. B. Darst, Louisiana State University, August, *On measures and measurability.*

H. T. David, University of Chicago, March, *The sample mean among the order statistics.*

L. C. Dean, Jr., Iowa State University, June, minor in Physics, *Nonlinear hyperbolic partial differential equation with small parameter.*

E. I. Deaton, University of Texas, August, *Solutions of a system of two nonlinear partial differential equations of the first order, with accessory boundary conditions.*

Philippe Dennery, Columbia University, *On conservation of probability in the Lee model.*

Betty Charles Detwiler, University of Kentucky, August, *A variational method for functions convex in the direction of the imaginary axis and related functions.*

R. S. DeZur, University of Oregon, June, *Homomorphisms on multiplicative semi-groups of continuous functions on a compact space.*

R. N. D'heedene, Harvard University, March, minor in Electronics, *Limit sets for nth order ordinary differential equations.*

M. R. Dorff, Iowa State University, June, *Large and small sample properties of estimators for a linear functional relation.*

A. C. Downing, Jr., University of Michigan, June, *On the convergence of steady state multiregion diffusion calculations.*

R. D. Driver, University of Minnesota, August, minor in Electrical Engineering, *Delay-differential equations and an application to a two-body problem of classical electrodynamics.*

S. D. Dubey, Michigan State University, June, *Contributions to statistical theory of life testing and reliability.*

J. R. Duffett, Virginia Polytechnic Institute, June, *System reliabilities from component reliabilities.*

Marguerite Elizabeth Dunton, University of Colorado, June, *Some contributions to the theory of diophantine equations.*

D. E. Dupree, Auburn University, August, *Existence and uniqueness of interpolating rational functions.*

A. L. Duquette, University of Colorado, August, *The analogue of the Pisot-Vijayaraghavan numbers in fields of formal power series.*

P. L. Duren, Massachusetts Institute of Technology, June, minor in Physics, *Spectral theory of a class of non-selfadjoint infinite matrix operators.*

T. A. Dwyer, Case Institute of Technology, June, minor in Physics, *Numerical analysis and nonlinear network problems.*

J. A. Dyer, University of Texas, August, minor in Physics, *On the consequences of momentum conservation laws in a gravitational theory of the Whitehead type.*

C. H. Edwards, Jr., University of Tennessee, December, *Concentric tori in the three-sphere.*

L. C. Eggan, University of Oregon, June, *On diophantine approximations.*

C. C. Elgot, University of Michigan, February, *Decision problems of finite automata design and related arithmetics.*

B. E. Ellison, University of Chicago, June, *A multivariate k-population classification problem.*

J. A. Ernest, University of Illinois, February, *Central intertwining numbers for representations of finite groups.*

D. J. Eustice, Purdue University, January, *Summability of orthogonal series.*

Leonard Evens, Harvard University, June, *The cohomology ring of a finite group.*

J. A. Ferling, University of Southern California, June, *A nonlinear eigenvalue problem for harmonic functions.*

K. M. Ferrin, University of California, Los Angeles, June, *Multiple decision procedures for normal populations.*

R. I. Fields, Virginia Polytechnic Institute, June, *Estimation with samples drawn from different but parametrically related distributions.*

A. M. Fink, Iowa State University, February, minor in History of Science, *Almost periodic points in topological transformation semigroups.*

Betty J. Isaacs Flehinger, Columbia University, November, *A general model for the reliability analysis of systems under various preventive maintenance policies.*

L. D. Fountain, University of Nebraska, August, minor in Physics, *The boundary value problem for an ordinary nonlinear differential equation of second order.*

Stanley Frank, University of Florida, February, minor in Education, *Certain cyclic involutory mappings on hyperspace surfaces.*

D. A. Freedman, Princeton University, October, *Mixtures of stochastic processes.*

M. L. Freimer, Harvard University, June, *Truncated policies in dynamic programming.*

A. H. Frey, Jr., University of Washington, June, *Studies on amenable semigroups.*

P. J. Freyd, Princeton University, June, *Functor theory.*

Yoichiro Fukuda, University of California, Los Angeles, June, *Estimation problems in inventory control.*

J. B. Garner, Auburn University, June, *Linear differential systems with two-point and three-point boundary conditions.*

E. D. Gaughan, University of Kansas, October, minor in Physics, *Generalized derivatives.*

D. W. Gaylor, North Carolina State College, May, *The construction and evaluation of some designs for the estimation of parameters in random models.*

C. W. Gear, University of Illinois, June, *Singular shock intersections in plane flow.*

Q. K. Ghori, University of British Columbia, May, minor in Physics, *On the equations of motion of mechanical systems subject to nonlinear nonholonomic constraints.*

J. D. Gilbert, Auburn University, August, *On subdirect products.*

S. C. Gitier, Princeton University, April, *Cohomology operations with bundles of coefficients.*

Orville Goering, Iowa State University, June, minor in Physics, *Dependence of the solution of a Goursat problem on the characteristic data.*

Ruth Zwerling Gold, Columbia University, June, *Inference about Markov chains with nonstationary transition probabilities.*

A. B. Gray, Jr., New Mexico State University, January, minor in Physics, *Infinite symmetric groups and monomial groups.*

D. A. Greenberg, Columbia University, *Theory of the hyperfine anomalies of deuterium, tritium and helium*$^{3+}$.

Martin Greendlinger, New York University, February, *Dehn's algorithm for the word problem.*

J. E. Grizzle, North Carolina State College, May, minor in Animal Industry, *Application of the logistic model to analyzing categorical data.*

B. I. Gross, University of Pennsylvania, February, *Groups of formal analytic transformations.*

Arnold Grudin, University of Colorado, June, minor in Philosophy, *Zeros of successive derivatives of entire functions.*

S. K. Gupta, Case Institute of Technology, June, *A theory of adjusting parameter-estimates in decision models.*

W. L. Hafley, North Carolina State College, May, minor in Forestry, *Some comparisons of sensitivities for two methods of measurement.*

Maurice Hanan, Carnegie Institute of Technology, June, *Oscillation criteria for third-order linear differential equations.*

V. R. Hancock, Tulane University, August, *Commutative Schreier extensions of semigroups.*

E. R. Hansen, Stanford University, October, *On Jacobi methods and Block-Jacobi methods for computing matrix eigenvalues.*

D. L. Hanson, Indiana University, June, *Contributions to decision theory, ergodic theory, and stochastic processes.*

J. E. Hanson, George Washington University, February, *On linear sequence spaces which permit omission and adjunction and have finite dimension modulo convergence.*

M. E. Harris, Harvard University, June, *Some results on a generalization of the character table of a finite group.*

C. A. Harvey, University of Minnesota, December, minor in Physics, *Existence of periodic solutions of the differential equation* $x''+g(x)=p(t)$.

M. P. Heble, Indiana University, June, *Linear estimation of regression coefficients; orthogonal matrix polynomials and application to multidimensional weakly stationary processes; interpolation and regression.*

Stevens Heckscher, Harvard University, March, *A characterization of certain Banach function spaces.*

Gertrude Ilse Heller, Johns Hopkins University, June, *On certain non-linear operators and partial differential equations.*

D. S. Henderson, Harvard University, June, *Logical designs for arithmetic units.*

S. W. Hess, Case Institute of Technology, June, *On research and development budgeting and project selection.*

P. D. Hill, Auburn University, June, *Limits of discrete groups.*

Heisuke Hironaka, Harvard University, June, *Theory of birational blowing-ups.*

C. R. Hobby, California Institute of Technology, June, *The derived series of a p-group.*

H. C. Hsieh, University of California, Berkeley, June, *The analysis of the effect of an obstacle on the electromagnetic field in a circular cylindrical wave guide.*

N.-C. Hsu, Washington University, January, *On automorphisms of a splitting extension* $G=(H, K; \phi)$.

T. C. Hu, Brown University, June, *Optimum design for structures of perfectly-plastic materials.*

B. E. Hubbard, University of Maryland, June, *Bounds for eigenvalues of the free and fixed membrane by finite difference methods.*

Taqdir Husain, Syracuse University, September, *On S-spaces and the open mapping theorem.*

S. Y. Husseini, Princeton University, June, *On the cohomology of exact sequences of compact groups.*

P. H. Hutcheson, University of Florida, August, minor in Chemistry, *The use of complex variables for solving certain elasticity problems involving intersecting boundaries.*

J. M. Irwin, University of Kansas, June, *High subgroups of Abelian torsion groups.*

J. E. Jackson, Virginia Polytechnic Institute, June, *Multivariate sequential procedures for testing means.*

Ronald Jacobowitz, Princeton University, June, *Hermitian forms over local fields.*

B. N. Jamison, University of California, Berkeley, September, *On ergodic theory of Markov operators.*

R. I. Jennrich, University of California, Los Angeles, *Analysis of variance in the general mixed model.*

H. L. Johnson, University of Minnesota, June, minor in Physics, *Quadratic versus linear dependence of solutions of certain linear partial differential equations.*

G. S. Jones, Jr., University of Cincinnati, August, *Asymptotic behavior and periodic solutions of a nonlinear differential-difference equation.*

W. L. Jones, Columbia University, November, *On conjugate functionals.*

C. L. Kaller, Purdue University, June, *A statistical approach to the study of genetic environmental interactions.*

H. M. Kamowitz, Brown University, June, *Cohomology groups of commutative Banach algebras.*

Julius Kane, New York University, June, minor in Physics, *Part* I. *An accurate boundary condition to replace transition conditions at dielectric-dielectric interfaces. Part* II. *Radio propagation past a dielectric interface.*

E. D. Kann, New York University, February, *Bonnet's theorem in two-dimensional G-spaces.*

J. E. Kelley, University of Michigan, June, *Characterization of the closed 2-cell and of the 2-sphere without assuming compactness.*

C. F. Kent, Massachusetts Institute of Technology, September, minor in Physics, *Algebraic structure of some groups of recursive permutations.*

S. A. Khabbaz, University of Kansas, June, *Theorems on Abelian groups.*

H.-C. Khare, McGill University, May, *Positron annihilation and scattering in helium.*

Masakiti Kinukawa, Northwestern University, June, *Fourier series.*

R. S. Kleber, State University of Iowa, June, *On the problem of minimum variance and maximum probability.*

Adam Kleppner, Harvard University, June, *Multipliers on Abelian groups.*

J. H. Klotz, University of California, Berkeley, September, *Nonparametric tests for scale.*

T. B. Knapp, Harvard University, March, *The relatively minimal models of a rational function field.*

A. G. Konheim, Cornell University, June, *Some properties of a class of finite trigonometric sums.*

F. J. Kosier, Michigan State University, September, *On a class of non-flexible algebras.*

Samuel Kotz, Cornell University, September, *Exponential bound for the probability of error in discrete memoryless channels.*

Kurt Kreith, University of California, Berkeley, June, *The spectrum of singular elliptic operators.*

B. M. Kurkjian, American University, June, *General theory for asymmetrical confounded factorial experiments.*

W. H. Lake, Catholic University of America, June, minor in Physics, *The numerical inversion of a particular class of matrices.*

L. H. Lange, University of Notre Dame, June, *Non-Euclidean Cercles de Remplissage and other analogues in the unit circle to classical theorems on entire functions.*

L. J. Lange, University of Colorado, August, *Divergence, convergence, and speed of convergence of continued fractions* $1+K(a_n/1)$.

R. P. Langlands, Yale University, June, *Semi-groups and representations of Lie groups.*

H. J. Larson, Iowa State University, August, minor in Industrial Engineering, *Sequential model building for prediction in regression analysis.*

A. T. Lauria, Purdue University, January, *One-dimensional retracts.*

Leon LeBlanc, University of Chicago, June, *Non-homogeneous and higher polyadic algebras.*

E. B. Lee, University of Minnesota, August, minor in Engineering, *Methods of optimum feedback control.*

L. M. Levine, New York University, June, *Diffraction by an elliptic cone.*

B. W. Levinger, New York University, June, *A generalization of the braid group.*

G. E. Lewis, New York University, February, *Two methods using power series for solving analytic non-characteristic initial value problems.*

J. A. Lindberg, Jr., University of Minnesota, August, *On the theory of algebraic extensions of a normed algebra.*

Seymour Lipschutz, New York University, February, *On the braid group.*

H.-C. Liu, University of Cincinnati, June, *Interpolation of entire functions.*

A. L. Liulevicius, University of Chicago, June, *The factorization of cyclic reduced powers by secondary cohomology operations.*

Stanley Locke, New York University, February, *A boundary layer theory of elastic plane stress.*

J. L. Locker, Auburn University, August, *A statistical analysis of the propagation of rounding error.*

C. A. Long, University of Illinois, June, minor in Physics, *Schwartz distributions analytic in a parameter.*

David Lubell, New York University, June, *Distribution functions for completely additive arithmetical functions on subsequences of the natural numbers.*

A. T. Lundell, Brown University, June, *Obstruction theory of principal fibre bundles.*

C. W. Lytle, New York University, February, *Differentiators for linear second order elliptic partial differential equations.*

E. B. McCue, Carnegie Institute of Technology, June, *Power characteristics of the control chart for number of defects, no standard given.*

J. E. McFarland, Oregon State College, June, *Iterative solution of nonlinear integral equations.*

R. A. McHaffey, Rutgers, The State University, June, *Structure theorems for a class of lattice ordered real Banach algebras.*

D. O. McKay, University of Buffalo, February, minor in Philosophy, *An extension of the Staudt-Clausen and Kummer congruences for the Bernoulli numbers of higher order.*

D. R. McMillan, Jr., University of Wisconsin, June, *On homologically trivial 3-manifolds.*

G. J. Maltese, Yale University, June, *Generalized convolution algebras and spectral representations.*

John Mariani, New York University, February, *Exponential solutions of linear differential equations of the second degree.*

P. H. Maserick, University of Maryland, August, *Half rings in linear spaces.*

T. K. Matthes, Columbia University, June, *Two-stage sampling procedures.*

J. G. May, University of Virginia, June, *Non-closed connected sets.*

W. G. May, University of Virginia, June, *Images of plane continua.*

J. W. Meux, University of Florida, August, minor in Education, *Orthogonal polynomial solutions of a class of fourth order linear differential equations.*

P. E. Miles, Yale University, June, *Order isomorphisms of B^* algebras.*

R. M. Mirman, Columbia University, *The dispersion relations for pion production in pion nuclear collisions.*

B. M. Mitchell, Brown University, June, *Homological tic tac toe.*

C. C. Moore, Harvard University, June, *Extensions and cohomology theory of locally compact topological groups.*

D. F. Morrison, Virginia Polytechnic Institute, June, *The life distribution and reliability of a system with spare components.*

J. F. Mount, University of California, Los Angeles, June, *Some applications of Schauder's theory to the calculus of variations and numerical analysis.*

K. R. Mount, University of California, Berkeley, June, *Characteristic classes of algebraic vector bundles.*

I. H. Mufti, The University of British Columbia, May, minor in Physics, *Stability in the large of autonomous systems of two differential equations.*

D. E. Myers, University of Illinois, February, *An imbedding space for Schwartz distributions.*

A.-A. K. Nafoosi, University of Colorado, June, *Representation of any large number as the sum of thirteen squares of positive integers in arithmetical progression.*

Sister Mary Redempta Nedumpilly, St. Louis University, June, minor in Physics, *On a generalized Feld series.*

L. W. Neustadt, New York University, February, *The moment problem and weak convergence in a Hilbert space.*

J. N. Newman, Massachusetts Institute of Technology, February, *Linearized theory for the motion of a thin ship in regular waves.*

H. H. Nickle, Columbia University, *Strong-coupling treatment of a charged scalar meson field interacting with a static extended source.*

R. N. van Norton, New York University, February, *The spectrum of a neutron transport operator.*

R. C. O'Neil, University of Chicago, December, *Fractional integration and Orlicz spaces.*

D. R. Ostberg, University of California, Berkeley, September, *Cohomology of groups and simple algebras.*

E. H. Ostrow, University of Chicago, August, *A theory of generalized Hilbert transforms.*

J. R. Padro, St. Louis University, June, *Extension and applications of cumulative characteristic functions.*

R. P. Pakshirajan, University of Oregon, June, *Regular measures and stochastic processes in topological groups.*

J. B. Pan, St. Louis University, June, minor in Physics, *On topological semigroups.*

Subramonier Parameswaran, University of Illinois, February, minor in Education, *Some theorems on the growth of partition functions.*

W. E. Parr, University of Maryland, June, minor in Physics, *Upper and lower bounds for the capacitance of the regular solids.*

C. W. Patty, University of Georgia, June, *Homotopy groups of certain deleted product spaces.*

K. M. Patwary, American University, June, *Error and non-error models in bio-assay.*

E. M. Paul, University of Illinois, October, minor in Philosophy, *Density in the light of probability theory.*

C. M. Pearcy, Jr., Rice University, June, *On the unitary equivalence of N-normal operators.*

J. M. Perry, University of Rochester, June, *Solution of boundary-value problems in arbitrary sectors by use of the double Laplace transform.*

M. W. Pownall, University of Pennsylvania, February, *An investigation of a conjecture of Goodman.*

Walter Pressman, New York University, June, *Evaluation of partition functions.*

W. E. Pruitt, Stanford University, June, *Bilateral birth and death processes.*

L. D. Pyle, Purdue University, June, *The generalized inverse in linear programming.*

D. E. A. Quade, University of North Carolina, June, *The asymptotic power of the Kolmogorov tests of goodness of fit.*

D. F. Rearick, California Institute of Technology, June, *Some visibility problems in point lattices.*

J. D. Reid, University of Washington, August, *Invariants of torsion free groups.*

J. I. Richards, Harvard University, June, *A classification of noncompact surfaces.*

Wyman Richardson, University of North Carolina, August, *Asymptotic methods of evaluating* $\int_a^\infty f(x)dx$.

Helen G. Murray Roberts, Boston University, June, *Two sequential tests against cyclic trend.*

Ruth Mabel Roberts, University of Pennsylvania, June, *On the solvability of a second order linear homogeneous differential equation.*

V. G. Robinson, Purdue University, June, *A study of mathematical models of epidemic disease distributions.*

Esther Rodlitz, New York University, February, *Deformation of Riemann surfaces.*

B. W. Romberg, University of Rochester, June, *The spaces* H_p *with* $0<p<1$.

H. M. Rosenblatt, George Washington University, February, *Multivariate experimental designs.*

Azriel Rosenfeld, Columbia University, *Specializations in differential algebra.*

Alan Ross, Iowa State University, February, *On two problems in sampling theory: unbiased ratio estimators and variance estimates in optimum sampling designs.*

K. A. Ross, University of Washington, March, *Studies in semigroups.*

Hugo Rossi, Massachusetts Institute of Technology, February, minor in Languages, *Maximality of algebras of holomorphic functions.*

R. D. Ryan, California Institute of Technology, June, *Fourier transforms of certain classes of integrable functions.*

M. J. Saadaldin, Duke University, September, *A generalized Lebesgue covering theorem.*

David Sachs, Illinois Institute of Technology, January, *Modulated and partition lattices.*

R. C. Sacksteder, Johns Hopkins University, February, minor in Physics, *Local and global properties of convex sets and hypersurfaces.*

A. A. Sagle, University of California, Los Angeles, August, *Malcev algebras.*

P. A. Scheinok, Indiana University, June, *The error on using the asymptotic variance and bias of spectrograph estimates for finite observation time.*

E. M. Scheuer, University of California, Los Angeles, June, *Simultaneous estimation for means or medians of dependent random variables without distribution assumptions.*

A. J. Schwartz, Wayne State University, August, *The geometric theory of non-compact transformation groups.*

Lorraine Schwartz, University of California, Berkeley, September, *Consistency of Bayes' procedures.*

Jack Segal, University of Georgia, June, *Inverse limit spaces.*

Tetsundo Sekiguchi, Oklahoma State University, August, *Representation theorems for summability operators and linear functionals on bounded sequences.*

D. W. H. Shale, University of Chicago, June, *On certain groups of operators on Hilbert space.*

W. T. Sharp, Princeton University, June, *Racah algebra and the contraction of groups.*

Aaron Siegel, Rutgers, The State University, January, *Summability C of series of surface spherical harmonics.*

Rajinder Singh, University of Illinois, June, minor in Economics, *Existence of bounded length confidence intervals.*

R. C. Singleton, Stanford University, October, *Steady state properties of selected inventory models.*

F. M. Sioson, University of California, Berkeley, June, *Contributions to the theory of primal and independent algebras.*

R. C. Smith, McGill University, May, *Central three-body nuclear forces.*

Ramaiyengar Sridharan, Columbia University, June, *Filtered algebras and representations of Lie algebras.*

S. R. Srivastava, Purdue University, June, *The power of an analysis of variance test procedure involving some preliminary tests for certain incompletely specified models.*

E. W. Stacy, University of North Carolina, June, *An estimate of correlation corrected for attenuation and its distribution.*

W. D. Stahlman, Brown University, June, *The astronomical tables of Codex Vaticanus Graecus* 1291.

J. G. Stampfli, University of Michigan, February, *On operators related to normal operators.*

H. M. Sternberg, University of Maryland, June, *The solution of the characteristic and the Cauchy boundary value problems for the Bianchi partial differential equation in n independent variables by a generalization of Riemann's method.*

H. R. Stevens, Duke University, June, minor in Philosophy, *Hurwitz product of sequences satisfying a generalized Kummer's congruence.*

S. H. Storey, McGill University, May, *The galvanomagnetic properties of some solids at low temperatures and high magnetic fields.*

S. L. Strack, Brown University, June, *Supersonic panel flutter of a finite cylinder.*

Charlotte Thomas Striebel, University of California, Berkeley, January, *Efficient estimation of regression parameters for certain second order stationary processes.*

Beauregard Stubblefield, University of Michigan, February, minor in Philosophy, *Some compact product spaces which cannot be imbedded in Euclidean n-space.*

Mudomo Sudigdomarto, University of Illinois, February, minor in Physics, *A representation theory for the Laplace transform of vector-valued functions.*

Shashikala Sukhatme, Michigan State University, September, *Asymptotic theory of some nonparametric tests.*

D. W. Swann, Stanford University, June, *Applications and extensions of the method of Wiener and Hopf for the solution of singular and non-singular integral and integro-differential equations.*

R. H. Szczarba, University of Chicago, August, *Homology of twisted cartesian products.*

R. J. Talham, Rensselaer Polytechnic Institute, January, *Directional receivers in correlation detection.*

S. G. Tellman, University of Washington, June, *Abelian groups with proper isomorphic quotient groups.*

Peter Terwey, Jr., Texas Agricultural and Mechanical College, May, minor in Physics, *Some aspects of gas dynamics in a closed region.*

R. C. Thompson, California Institute of Technology, June, *Commutators in the special and general linear groups.*

L. J. Tick, Columbia University, September, minor in Sociology, *Contributions to theory and application of random processes in fluid mechanics.*

T. W. Ting, Indiana University, June, *Fracture of closed circular pipes under internal pressure and axial tension.*

R. J. Troyer, Indiana University, September, *Multilinear algebra in Abelian categories.*

E. J. Tully, Jr., Tulane University, June, *Representation of a semigroup by transformations of a set.*

R. L. Van de Wetering, Stanford University, October, *On the motions of particles in Euclidean and non-Euclidean spaces under certain conservative force fields.*

F. S. Van Vleck, University of Minnesota, August, *Bifurcation of an invariant manifold from a periodic solution of a differential system.*

K. Varadarajan, Columbia University, June, *Dimension, category and $K(\pi, n)$ spaces.*

H. M. Wadsworth, Jr., Western Reserve University, June, *A stochastic theory of documentation.*

J.-K. Wang, Stanford University, January, *Multipliers of commutative Banach algebras.*

M. T. Wasan, University of Illinois, June, *Sequential estimation of a binomial parameter.*

J. R. Webb, University of Texas, August, minor in Physics, *A Hellinger integral representation for bounded linear functionals.*

J. T. Webster, North Carolina State College, July, *A decision procedure for the inclusion of an independent variate in a linear estimator.*

D. F. Wehn, Yale University, June, *Limit distributions on Lie groups.*

P. M. Weichsel, California Institute of Technology, June, *A decomposition theory for finite groups with applications to p-groups.*

F. W. Weiler, Ohio State University, June, *On the T-Jacobian.*

G. G. Weill, University of California, Los Angeles, January, *Reproducing kernels and orthogonal kernels for analytic differentials on Riemann surfaces.*

R. R. Welland, Purdue University, June, *Local integrability in σ-finite measure spaces and Köthe spaces.*

Roy Westwick, University of British Columbia, May, minor in Physics, *Linear transformations on Grassmann product spaces.*

T. A. Willke, Ohio State University, August, *A class of multivariate rank statistics.*

J. C. Wilson, Case Institute of Technology, June, *Analysis of concepts of differentiability on algebras.*

D. M. G. Wishart, Princeton University, October, *Augmentation techniques in the theory of queues.*

W. W. Wolman, University of Rochester, June, *A problem in the design and analysis of experiments with correlated observations.*

T.-C. Woo, Brown University, June, *Fundamental solutions for small deformations superposed on finite biaxial extension of an elastic body.*

W. B. Woolf, University of Michigan, February, *Radial cluster sets and the distribution of values of meromorphic functions.*

J. Z.-T. Yao, University of Chicago, June, *Moore-Catan theorems and Leray-Serre theorem.*

Bohyun Yim, Brown University, June, *Supersonic flow past double wedge wings with variable thickness.*

N. D. Ylvisaker, Stanford University, January, *On time series analysis and reproducing kernel spaces.*

P. J. Zwier, Purdue University, August, *Homology and cohomology from rings of functions.*

Journals Published by the

American Mathematical Society

Bulletin of the American Mathematical Society

This journal is the official organ of the Society. It reports official acts of the Society and the details of its meetings. It contains some of the officially invited addresses presented before the Society, reviews of advanced mathematical books, research problems and a department of research announcements.

The subscription prize is $7.00 per annual volume of six numbers.

Research Problems and Invited Addresses offered for publication should be sent to Walter Rudin, Department of Mathematics, University of Wisconsin, Madison, Wisconsin; Book Reviews to Felix Browder, Department of Mathematics, Yale University, New Haven, Connecticut. Research Announcements offered for publication should be sent to some member of the Council of the Society, and communicated by him to E. E. Moise, Department of Mathematics, Harvard University, Cambridge 38, Massachusetts. All other communications to the editors should be sent to the Managing Editor, E. E. Moise.

The members of the Council for 1961 are: P. T. Bateman, E. G. Begle, Lipman Bers, R. P. Boas, H. W. Bode, H. F. Bohnenblust, Raoul Bott, Felix Browder, R. H. Bruck, R. C. Buck, M. M. Day, Jean Dieudonné, J. L. Doob, A. Erdelyi, G. E. Forsythe, P. R. Garabedian, A. M. Gleason, J. W. Green, P. R. Halmos, G. A. Hedlund, M. R. Hestenes, Edwin Hewitt, Einar Hille, G. P. Hochschild, G. B. Huff, Nathan Jacobson, Michel Loève, E. J. McShane, W. S. Massey, A. E. Meder, Jr., E. E. Moise, Deane Montgomery, L. J. Paige, R. S. Phillips, R. S. Pierce, Everett Pitcher, G. de B. Robinson, Alex Rosenberg, Walter Rudin, I. J. Schoenberg, I. M. Singer, E. H. Spanier, C. B. Tompkins, A. W. Tucker, S. M. Ulam, J. V. Wehausen, J. W. T. Youngs, O. Zariski, Daniel Zelinsky, Leo Zippin, Antoni Zygmund.

Proceedings of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics and is devoted principally to the publication of original papers of moderate length. A department called Mathematical Pearls was established in 1961. The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

The subscription price is $11.00 per annual volume of six numbers.

Papers in algebra and number theory should be sent to Alex Rosenberg, Lunt Building, Northwestern University, Evanston, Illinois; in probability, real variables, logic and foundations to P. R. Halmos, Eckhart Hall, University of Chicago, Chicago 37, Illinois; in abstract analysis to either P. R. Halmos or Alex Rosenberg; in geometry and topology to E. H. Spanier, Department of Mathematics, University of California, Berkeley, California; in other branches of analysis, applied mathematics, and all other fields to R. P. Boas, Lunt Building, Northwestern University, Evanston, Illinois. All other communications to the editors should be addressed to the Managing Editor, P. R. Halmos.

Transactions of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics, and includes in general longer papers than the Proceedings.

Four volumes of three numbers each will be published in 1960. The subscription price is $8.00 per volume.

Papers in analysis and applied mathematics should be sent to Lipman Bers, Institute of Mathematical Sciences, New York University, New York, New York; in topology to W. S. Massey, Department of Mathematics, Yale University, Box 2155, Yale Station, New Haven, Connecticut; in algebra, number theory, and logic to Daniel Zelinsky, Department of Mathematics, University of California, Berkeley 4, California; in geometry and abstract analysis to I. M. Singer, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Massachusetts; in statistics and probability to Michel Loève, Statistics Department, University of California, Berkeley, California. All other communications to the editors should be addressed to the Managing Editor, Lipman Bers.